Perturbative path-integral study of active- and passive-tracer diffusion in fluctuating fields

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We study the effective diffusion constant of a Brownian particle linearly coupled to a thermally fluctuating scalar field. We use a path-integral method to compute the effective diffusion coefficient perturbatively to lowest order in the coupling constant. This method can be applied to cases where the field is affected by the particle (an active tracer) and cases where the tracer is passive. Our results are applicable to a wide range of physical problems, from a protein diffusing in a membrane to the dispersion of a passive tracer in a random potential. In the case of passive diffusion in a scalar field, we show that the coupling to the field can, in some cases, speed up the diffusion corresponding to a form of stochastic resonance. Our results on passive diffusion are also confirmed via a perturbative calculation of the probability density function of the particle in a Fokker-Planck formulation of the problem. Numerical simulations on simplified systems corroborate our results.

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I. INTRODUCTION

Diffusion in a quenched random medium is a problem which has been extensively investigated [1,2]. An important physical problem is to understand how the long-time transport properties of a Brownian particle, notably its effective diffusion constant, are modified with respect to those of a homogeneous medium. Two types of problems with quenched disorder have been extensively studied. The first is diffusion in a medium where the local diffusion constant depends on spatial position and is taken to have some statistical distribution, and the second is for a particle that is advected by a quenched random velocity field. The case where the random velocity field is derived from a potential is of importance as a toy model for spin glasses and glasses as it can exhibit a static spin glasslike transition [3,4] and a dynamical structural glasslike transition [5–7]. Variants of the toy model of diffusion in a random potential can exhibit a transition in their transport properties, notably diffusion, which is normal in the high-temperature phase and becomes subdiffusive in the low-temperature phase [1,8–10]. The onset of the subdiffusive regime is signaled by the vanishing of the late-time diffusion constant. It should also be mentioned that the above class of problems can also be related the problem of evaluating the macroscopic (effective) electrical properties, such as conductivities or dielectric constants, of random conductors or dielectrics [2].

The case of diffusion in dynamically evolving random media has perhaps received less attention. The most widely studied problem of a time-dependent nature is for a particle diffusing in a turbulent flow [11,12], and the particle here is a passive tracer and has no effect on the flow field. Another example is for a protein diffusing in a membrane, where the protein is subject to a force generated by membrane curvature or composition, and these quantities themselves fluctuate. However, in this case the membrane is in general affected by the protein [13]. Naively one might expect that protein diffusion is speeded up by coupling to height or composition fluctuations in a membrane; however, this is not the case: The feedback of the protein on the membrane configuration actually slows the protein diffusion with respect to a nonfluctuating homogeneous one. A general question that arises in these sorts of problems is: When does the fluctuating field speed up the diffusion of a tracer and when does it slow it down? An important question, which has received much recent attention, is how the diffusion constant of a protein depends on its size. The classic hydrodynamic computation of Saffmann and Delbrück [14] treats the protein as a solid cylinder in an incompressible layer of fluid (the lipid bilayer) sandwiched between another fluid (the water). In this formalism the protein diffusivity shows a weak logarithmic dependence on the cylinder radius. However, the validity of this result has been called into question experimentally where a stronger dependence is reported [15]. A number of theoretical studies have suggested that a protein's diffusion could be modified by coupling to local membrane properties, such as composition and curvature, that are not taken into account in a purely hydrodynamic model for a membrane [13,16–22]. One should bear in mind that protein coupling to local geometry and composition has also been postulated as a mechanism for interprotein interactions in biological membranes [23,24].

In this paper we will consider a particle, whose position is denoted by $\boldsymbol{x}(t)$, diffusing in a scalar field $h\psi$, where h is a parameter controlling the coupling of the field to the particle. Thus the particle drift is given by $\boldsymbol{u} = -\kappa_x h \nabla\psi$, where κ_x denotes the bare diffusion constant of the particle. The dynamics of both the particle and the field are overdamped and stochastic. Our aim is to evaluate how the effective diffusion coefficient for the particle is modified by its coupling to the field. A path-integral approach allows us to perform a perturbative computation if the coupling constant between the particle and the field is small for a wide range of physically different problems. Within this formalism we recover some previous results on the diffusion of active tracers where the whole system (particle plus field) obeys detailed balance [22]. However, the method also allows us to analyze passive diffusion and a continuum of intermediate models with varying degrees of feedback of the particle on the field and for thermal fluctuations on the particle and field that are not necessarily at the same temperature. This intermediate range of models could apply to the study of tracers in nonthermally driven fields and have applications to active colloidal systems [25], swimmers [26], or systems where the tracer is heated with an external heat source such as a laser [27]. The basic method may also be useful for studying systems where the field itself is nonthermally driven, for instance, lipid bilayers, where the field is driven by an electric field [28].

The formalism also allows us to see how the relative time scales between the tracer and fluctuating field affect the effective diffusion constant of the tracer.

Our results for the effective diffusion constant show very different effects for active and passive diffusion, and diffusion in a quenched potential. In pure active diffusion, where the stochastic equations of motion obey detailed balance, the diffusion is always slowed. For passive diffusion, if the field evolution is slow, the particle slows, as if it was evolving in a quenched potential [2]. As the field evolution is speeded up, the diffusion constant increases and then reaches a maximum, in a manner reminiscent of stochastic resonance [29–32]. As the field evolution rate increases still further, the diffusion constant diminishes and eventually reaches the bare value it would have in the absence of coupling to a field. The results derived via the path-integral formalism for this case are also derived via perturbation theory on the corresponding time-dependent Fokker-Planck equation.

Finally we demonstrate the analytically predicted effects by numerical simulation of simple models where the fluctuating field has a small number of Fourier modes.

II. THE MODEL

To start, we define the general class of models of interest to us in this paper. First, we consider the dynamics of a Langevin particle whose position is denoted by $\boldsymbol{x}(t)$ diffusing in a d-dimensional space with a linear coupling to a fluctuating Gaussian field, as introduced in Ref. [22]. The overall energy (Hamiltonian) for the system is

$$H = \frac{1}{2}\int \phi(\boldsymbol{x})\Delta\phi(\boldsymbol{x})\,d\boldsymbol{x} - hK\phi[\boldsymbol{x}(t)], \tag{1}$$

where the first term is the quadratic energy of a free scalar field, and the second corresponds to the tracer seeing an effective potential $h\psi = -hK\phi$. We will take Δ and K to be self-adjoint operators. For two operators $A(\boldsymbol{x},\boldsymbol{y})$ and $B(\boldsymbol{x},\boldsymbol{y})$ we will denote by $AB(\boldsymbol{x},\boldsymbol{y})$ their composition as operators: $(AB)(\boldsymbol{x},\boldsymbol{y}) = \int A(\boldsymbol{x},z)B(z,\boldsymbol{y})\,dz$.

For a system obeying detailed balance (and whose equilibrium state is thus given by the Gibbs-Boltzmann distribution) the particle evolves according to

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= -\kappa_x\frac{\delta H}{\delta \boldsymbol{x}} + \sqrt{\kappa_x}\boldsymbol{\eta}(t)\\ &= h\kappa_x\nabla K\phi[\boldsymbol{x}(t)] + \sqrt{\kappa_x}\boldsymbol{\eta}(t),\end{aligned} \tag{2}$$

where the noise term is Gaussian with mean zero and correlation function

$$\langle \boldsymbol{\eta}(t)\boldsymbol{\eta}(s)^T\rangle = 2T\delta(t-s)\mathbf{1}, \tag{3}$$

where T is the temperature of the system.

In the absence of a coupling between the field and the particle ($h=0$), the particle diffuses normally, and the mean-squared displacement at large times behaves as

$$\langle[\boldsymbol{x}(t)-\boldsymbol{x}(0)]^2\rangle \underset{t\to\infty}{\sim} 2dT\kappa_x t = 2dD_x t, \tag{4}$$

where d is the spatial dimension and $D_x = T\kappa_x$ is the bare diffusion constant.

We take a general dissipative dynamics for the field [33]:

$$\begin{aligned}\dot{\phi}(\boldsymbol{x},t) &= -\kappa_\phi R\frac{\delta H}{\delta\phi(\boldsymbol{x})} + \sqrt{\kappa_\phi}\xi(\boldsymbol{x},t)\\ &= -\kappa_\phi R\Delta\phi(\boldsymbol{x}) + h\kappa_\phi RK[\boldsymbol{x}-\boldsymbol{x}(t)] + \sqrt{\kappa_\phi}\xi(\boldsymbol{x},t),\end{aligned} \tag{5}$$

where R is a self-adjoint dynamical operator and ξ is a Gaussian noise of zero mean that is uncorrelated in time. In order for the field to equilibrate to the Gibbs-Boltzmann distribution, the correlation function of this noise must be

$$\langle\xi(\boldsymbol{x},t)\xi(\boldsymbol{y},s)\rangle = 2TR(\boldsymbol{x}-\boldsymbol{y})\delta(t-s). \tag{6}$$

The effective diffusion constant for the particle is defined via

$$\langle[\boldsymbol{x}(t)-\boldsymbol{x}(0)]^2\rangle \underset{t\to\infty}{\sim} 2dT\kappa_e t = 2dD_e t, \tag{7}$$

where D_e is the effective or *late time* diffusion constant.

Our model applies to many systems: A point magnetic field of magnitude h diffusing in a *Gaussian ferromagnet* can be modeled with $\Delta(\boldsymbol{x},\boldsymbol{y}) = (-\nabla_x^2 + m^2)\delta(\boldsymbol{x}-\boldsymbol{y})$, $K(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})$, and $R(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})$ for model A dynamics or $R(\boldsymbol{x},\boldsymbol{y}) = -\nabla_x^2\delta(\boldsymbol{x}-\boldsymbol{y})$ for model B conserved dynamics [33]. Here the field ϕ can correspond to the fluctuations of a range of order parameters in a lipid membrane, in the Gaussian approximation, where the fluctuations are weak [20,21]. The order parameter in question could be compositional fluctuations above the demixing transition in lipid bilayers or binary fluids. Here a delta function form of the coupling K would correspond to the tracer having a preference for one of the lipid phases or preferential wetting for one of the phases of a binary fluid. As well as compositional fluctuations, ϕ could also correspond to fluctuations in local lipid ordering (gel and liquid phases) and possibly orientational order in the lipid tails. It could also correspond to local membrane thickness in the case where the protein induces a local hydrophobic mismatch and locally alters the thickness of the bilayer.

To model a lipid membrane where the field represents the height fluctuations and the particle is a protein coupled to membrane curvature, we may take the Helfrich Hamiltonian [34] $\Delta(\boldsymbol{x},\boldsymbol{y}) = (\kappa\nabla_x^4 - \sigma\nabla_x^2)\delta(\boldsymbol{x}-\boldsymbol{y})$, $K(\boldsymbol{x},\boldsymbol{y}) = -\nabla_x^2\delta(\boldsymbol{x}-\boldsymbol{y})$, and R is given by its Fourier transform, $\tilde{R}(\boldsymbol{k}) = 1/4\eta|\boldsymbol{k}|$, where η is the viscosity of the solvent surrounding the membrane [35,36].

The computations below can be made a little more general. As mentioned in the introduction there are a number of physical cases of nonequilibrium systems where one is not restricted to stochastic dynamics obeying detailed balance and, for instance, the coupling between the particle and the field may be taken as nonsymmetric: Instead of (5), we will consider

$$\dot{\phi}(\boldsymbol{x},t) = -\kappa_\phi R\Delta\phi(\boldsymbol{x}) + \zeta h\kappa_\phi RK[\boldsymbol{x}-\boldsymbol{x}(t)] + \sqrt{\kappa_\phi}\xi(\boldsymbol{x},t), \tag{8}$$

where we have introduced the parameter ζ. The case of stochastic dynamics with detailed balance is recovered for $\zeta = 1$. We note that Eq. (2) has been extensively studied in the case where the field ϕ evolves independently of the particle position, which corresponds to $\zeta = 0$. This problem is referred to as the advection diffusion of a passive scalar (the particle concentration) in a fluctuating field ϕ. It had been suggested that this form can be used to approximate the diffusion of an active tracer particle in Refs. [16,18], for the case of a protein weakly coupled to membrane curvature. In this approximation it was found that the effect of the field fluctuations could be to increase the diffusivity of the tracer particle with respect to that obtained when it is not coupled to the fluctuating field ($h = 0$). However, the numerical simulations of Ref. [13] where the effect of the particle position on the field is taken into account showed that the diffusion is reduced with respect to the case $h = 0$, in agreement with later analytical studies [19,22].

Also as previously mentioned, a further generalization can be made to our model: Since, in a nonequilibrium system, the particle and the field are not necessarily driven by the same thermal bath, they can experience different temperatures. In this case the temperatures appearing in the correlation function of the noises in Eqs. (3) and (6) will be denoted, respectively, $T_{\boldsymbol{x}}$ and T_ϕ.

III. EFFECTIVE DIFFUSION EQUATION AND PATH-INTEGRAL FORMALISM

A. Effective diffusion equation

Our aim is to study the average value of the mean-squared displacement of the particle, we thus integrate the dynamical equation of field Eq. (8), assuming without loss of generality that the field $\phi = 0$ at time $t = 0$. This gives

$$\phi(\boldsymbol{x},t) = \int_{-\infty}^{t} e^{-\kappa_\phi(t-s)R\Delta} \times \{\zeta h\kappa_\phi RK[\boldsymbol{x}-\boldsymbol{x}(s)] + \sqrt{\kappa_\phi}\xi(\boldsymbol{x},s)\}\,ds. \tag{9}$$

Using this result in Eq. (2) we obtain the effective diffusion equation for the particle:

$$\dot{\boldsymbol{x}}(t) = h\kappa_{\boldsymbol{x}}\nabla K \int_{-\infty}^{t} e^{-\kappa_\phi(t-s)R\Delta}\{\zeta h\kappa_\phi RK[\boldsymbol{x}(t)-\boldsymbol{x}(s)] + \sqrt{\kappa_\phi}\xi[\boldsymbol{x}(t),s]\}\,ds + \sqrt{\kappa_{\boldsymbol{x}}}\boldsymbol{\eta}(t). \tag{10}$$

The right-hand term can be split into two parts: a deterministic part depending only on the particle trajectory, and a stochastic part that depends on the noise driving the field and on the particle trajectory:

$$\dot{\boldsymbol{x}}(t) = \sqrt{\kappa_{\boldsymbol{x}}}\boldsymbol{\eta}(t) + \int_{-\infty}^{t} \boldsymbol{F}[\boldsymbol{x}(t)-\boldsymbol{x}(s),t-s]\,ds + \boldsymbol{\Xi}[\boldsymbol{x}(t),t], \tag{11}$$

where $\boldsymbol{\Xi}$ is a Gaussian noise dependent on the position in space and time, with correlation function

$$\langle \boldsymbol{\Xi}(\boldsymbol{x},t)\boldsymbol{\Xi}(\boldsymbol{y},s)^T\rangle = T\boldsymbol{G}(\boldsymbol{x}-\boldsymbol{y},t-s), \tag{12}$$

and we have introduced the functions

$$\boldsymbol{F}(\boldsymbol{x},u) = \zeta h^2\kappa_{\boldsymbol{x}}\kappa_\phi \nabla K e^{-\kappa_\phi uR\Delta}RK(\boldsymbol{x}), \tag{13}$$

$$\boldsymbol{G}(\boldsymbol{x},u) = -h^2\kappa_{\boldsymbol{x}}^2\nabla\nabla^T K^2 e^{-\kappa_\phi|u|R\Delta}\Delta^{-1}(\boldsymbol{x}). \tag{14}$$

We introduce the above functions for two reasons: to provide more compact notation for the switch to the path-integral formalism, and show explicitly which part of the following computation is general and does not depend on the expressions of $\boldsymbol{F}$ and $\boldsymbol{G}$. Indeed, for a different choice of functions, (11) and (12) define another model, which may be analyzed using the method presented here. We will need explicit expressions for $\boldsymbol{F}$ and $\boldsymbol{G}$; in particular, their $\boldsymbol{x}$ dependence is rather obscure. Fourier transforming allows us to write them as a sum of functions with a completely explicit $\boldsymbol{x}$ dependence:

$$\boldsymbol{F}(\boldsymbol{x},u) = \zeta h^2\kappa_{\boldsymbol{x}}\kappa_\phi \int \frac{d^d\boldsymbol{k}}{(2\pi)^d} i\boldsymbol{k}e^{-\kappa_\phi u\tilde{R}(\boldsymbol{k})\tilde{\Delta}(\boldsymbol{k})}\tilde{R}(\boldsymbol{k})\tilde{K}^2(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \int \frac{d^d\boldsymbol{k}}{(2\pi)^d}\boldsymbol{F}_{\boldsymbol{k}}(\boldsymbol{x},u) \tag{15}$$

and

$$\boldsymbol{G}(\boldsymbol{x},u) = h^2\kappa_{\boldsymbol{x}}^2 \int \frac{d^d\boldsymbol{k}}{(2\pi)^d}\boldsymbol{k}\boldsymbol{k}^T \frac{e^{-\kappa_\phi|u|\tilde{R}(\boldsymbol{k})\tilde{\Delta}(\boldsymbol{k})}\tilde{K}^2(\boldsymbol{k})}{\tilde{\Delta}(\boldsymbol{k})}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \int \frac{d^d\boldsymbol{k}}{(2\pi)^d}\boldsymbol{G}_{\boldsymbol{k}}(\boldsymbol{x},u). \tag{16}$$

B. Path-integral formulation

We now turn to the path-integral formalism; the first steps are analogous to those described for general stochastic dynamics in Refs. [37–39] and [40] for transport by a time-dependent incompressible velocity field. The *partition function* for this system is

$$Z = \int \prod_t \delta\left\{\dot{\boldsymbol{x}}(t) - \sqrt{\kappa_{\boldsymbol{x}}}\boldsymbol{\eta}(t) - \int_{-\infty}^{t}\boldsymbol{F}[\boldsymbol{x}(t)-\boldsymbol{x}(s),t-s]\,ds - \boldsymbol{\Xi}(\boldsymbol{x}(t),t)\right\} P[\boldsymbol{\eta}]Q[\boldsymbol{\Xi}][d\boldsymbol{x}][d\boldsymbol{\eta}][d\boldsymbol{\Xi}], \tag{17}$$

where P and Q are the functional Gaussian weight for the noises. Note that we should in principle include a Jacobian in the above expression. However, if we use the Itô convention, the Jacobian term is constant and independent of the path. This is because the causality of the equation of motion and the use of the Itô calculus mean that the transformation matrix is triangular with diagonal terms that are constant [39]. We then use a functional integral description of the δ function, introducing the vector field $\boldsymbol{p}$,

$$Z = \int \exp\left(i\int \boldsymbol{p}(t)\cdot\left\{\dot{\boldsymbol{x}}(t) - \sqrt{\kappa_{\boldsymbol{x}}}\boldsymbol{\eta}(t) - \int_{-\infty}^{t}\boldsymbol{F}[\boldsymbol{x}(t) - \boldsymbol{x}(s),t-s]\,ds - \boldsymbol{\Xi}(\boldsymbol{x}(t),t]\right\}dt\right) \times P[\boldsymbol{\eta}]Q[\boldsymbol{\Xi}][d\boldsymbol{x}][d\boldsymbol{p}][d\boldsymbol{\eta}][d\boldsymbol{\Xi}]. \tag{18}$$

Then we use the standard result that, for a Gaussian random variable of zero mean u, $\langle\exp(au)\rangle = \exp(a^2\langle u^2\rangle/2)$ to perform the integration over the noises to obtain

$$Z = \int \exp(-S[\boldsymbol{x},\boldsymbol{p}])[d\boldsymbol{x}][d\boldsymbol{p}], \tag{19}$$

where the action is given by

$$\begin{aligned}S[\boldsymbol{x},\boldsymbol{p}] = -i\int \boldsymbol{p}(t)\cdot\bigg\{\dot{\boldsymbol{x}}(t) - \int_{-\infty}^{t} \boldsymbol{F}[\boldsymbol{x}(t)\\ - \boldsymbol{x}(s),t-s]\,ds\bigg\}dt + D_{\boldsymbol{x}}\int|\boldsymbol{p}(t)|^2\,dt\\ + \frac{T_\phi}{2}\int \boldsymbol{p}^T(t)\boldsymbol{G}(\boldsymbol{x}(t)-\boldsymbol{x}(s),t-s)\boldsymbol{p}(s)\,dt\,ds.\end{aligned} \tag{20}$$

This action is the sum of the action of the pure Brownian motion

$$S_0[\boldsymbol{x},\boldsymbol{p}] = -i\int \boldsymbol{p}(t)\cdot\dot{\boldsymbol{x}}(t)\,dt + D_{\boldsymbol{x}}\int|\boldsymbol{p}(t)|^2\,dt \tag{21}$$

and the action of the interaction

$$\begin{aligned}S_{\text{int}}[\boldsymbol{x},\boldsymbol{p}] = i\int \boldsymbol{p}(t)\cdot\boldsymbol{F}[\boldsymbol{x}(t)-\boldsymbol{x}(s),t-s]\theta(t-s)\,dt\,ds\\ + \frac{T_\phi}{2}\int \boldsymbol{p}^T(t)\boldsymbol{G}[\boldsymbol{x}(t)-\boldsymbol{x}(s),t-s]\boldsymbol{p}(s)\,dt\,ds.\end{aligned} \tag{22}$$

Since $\boldsymbol{G}(-\boldsymbol{x},-u) = \boldsymbol{G}(\boldsymbol{x},u)$, we can write this integral only with times satisfying $t \geqslant s$, which will be convenient for the ensuing calculations:

$$\begin{aligned}&S_{\text{int}}[\boldsymbol{x},\boldsymbol{p}]\\ &= i\int \boldsymbol{p}(t)\cdot\boldsymbol{F}[\boldsymbol{x}(t)-\boldsymbol{x}(s),t-s]\theta(t-s)\,dt\,ds\\ &\quad + T_\phi\int \boldsymbol{p}^T(t)\boldsymbol{G}[\boldsymbol{x}(t)-\boldsymbol{x}(s),t-s]\boldsymbol{p}(s)\theta(t-s)\,dt\,ds.\end{aligned} \tag{23}$$

C. Computing averages with the free action

We will need to compute averages with the free action S_0. Since it is quadratic in $\boldsymbol{x}$ and $\boldsymbol{p}$, we need just the one- and two-point correlation functions. Moreover, the position of the particle is relevant only with respect to its position at, say, $t = 0$. In order to keep compact notations, we define

$$\boldsymbol{x}_0(t) = \boldsymbol{x}(t) - \boldsymbol{x}(0). \tag{24}$$

The correlation functions required are $\langle\boldsymbol{x}_0(t)\rangle_0$, $\langle\boldsymbol{p}(t)\rangle_0$, $\langle\boldsymbol{p}(t)\boldsymbol{p}(s)^T\rangle_0$, $\langle\boldsymbol{x}_0(t)\boldsymbol{p}(s)^T\rangle_0$, and $\langle\boldsymbol{x}_0(t)\boldsymbol{x}_0(s)^T\rangle_0$.

Using the symmetry of S_0, we have immediately

$$\langle\boldsymbol{x}_0(t)\rangle_0 = \boldsymbol{0}, \tag{25}$$

$$\langle\boldsymbol{p}(t)\rangle_0 = \boldsymbol{0}. \tag{26}$$

To obtain the two-point correlation functions, we use the fact that the (functional) integral of a total (functional) derivative is zero, for example,

$$\begin{aligned}\boldsymbol{0} &= \int \frac{\delta}{\delta\boldsymbol{x}(s)}[\boldsymbol{p}(t)e^{-S_0}][d\boldsymbol{x}][d\boldsymbol{p}]\\ &= \int \boldsymbol{p}(t)\dot{\boldsymbol{p}}(s)^T e^{-S_0}[d\boldsymbol{x}][d\boldsymbol{p}];\end{aligned} \tag{27}$$

dividing each side by Z_0, we obtain $\langle\boldsymbol{p}(t)\dot{\boldsymbol{p}}(s)^T\rangle_0 = \boldsymbol{0}$: $\langle\boldsymbol{p}(t)\boldsymbol{p}(s)^T\rangle_0$ is a constant. This constant must be zero, because the action S_0 does not correlate $\boldsymbol{p}$ at different times: $\boldsymbol{p}(t)$ are all independent. We thus have the first correlator

$$\langle\boldsymbol{p}(t)\boldsymbol{p}(s)^T\rangle_0 = \boldsymbol{0}. \tag{28}$$

We use the same technique for the other correlators:

$$\begin{aligned}\boldsymbol{0} &= \int \frac{\delta}{\delta\boldsymbol{p}(s)}[\boldsymbol{p}(t)e^{-S_0}][d\boldsymbol{x}][d\boldsymbol{p}]\\ &= \int\{\delta(t-s)\boldsymbol{1} + \boldsymbol{p}(t)[i\dot{\boldsymbol{x}}(s)^T - 2D_{\boldsymbol{x}}\boldsymbol{p}(s)^T]\}e^{-S_0}[d\boldsymbol{x}][d\boldsymbol{p}],\end{aligned} \tag{29}$$

which gives, after dividing by Z_0 and integrating over $s \in [0,t]$,

$$\langle\boldsymbol{x}_0(t)\boldsymbol{p}(s)^T\rangle_0 = i\chi_{[0,t)}(s)\boldsymbol{1}, \tag{30}$$

where $\chi_A(s)$ is the characteristic function of the set A (equal to 1 if the argument is in A and zero elsewhere). Finally the identity,

$$\begin{aligned}\boldsymbol{0} &= \int \frac{\delta}{\delta\boldsymbol{p}(s)}(\boldsymbol{x}_0(t)e^{-S_0})[d\boldsymbol{x}][d\boldsymbol{p}]\\ &= \int \boldsymbol{x}_0(t)[i\dot{\boldsymbol{x}}(s)^T - 2D_{\boldsymbol{x}}\boldsymbol{p}(s)^T]e^{-S_0}[d\boldsymbol{x}][d\boldsymbol{p}],\end{aligned} \tag{31}$$

leads to

$$\langle\boldsymbol{x}_0(t)\boldsymbol{x}_0(s)^T\rangle_0 = 2D_{\boldsymbol{x}}L([0,t]\cap[0,s])\boldsymbol{1}, \tag{32}$$

where $L(I)$ is the length of the interval $I \subset \mathbb{R}$; this is, of course, the standard result for free Brownian motion $\langle\boldsymbol{x}_0(t)\boldsymbol{x}_0(s)^T\rangle_0 = 2D_{\boldsymbol{x}}\min(t,s)$ if $t,s \geqslant 0$.

IV. PERTURBATIVE CALCULATION OF THE EFFECTIVE DIFFUSION CONSTANT

A. Derivation of the general result

Here we should go back to our aim of computing the effective diffusion constant D_e. To do this, we have to evaluate $\langle\boldsymbol{x}_0(t_f)^2\rangle$ at a large time t_f. Since we do not know how to compute averages with the action S, we use a perturbation expansion in terms of averages over S_0, which will be denoted $\langle\cdots\rangle_0$:

$$\langle\boldsymbol{x}_0(t_f)^2\rangle = \frac{\langle\boldsymbol{x}_0(t_f)^2\exp(-S_{\text{int}}[\boldsymbol{x},\boldsymbol{p}])\rangle_0}{\langle\exp(-S_{\text{int}}[\boldsymbol{x},\boldsymbol{p}])\rangle_0}. \tag{33}$$

Averages appearing here are not easy to compute, but it is easy to compute the first nontrivial term in expansion in the interaction action $S_{\rm int}$. To do this, we just expand the exponential functions:

$$\langle \boldsymbol{x}_0(t_f)^2\rangle \simeq \frac{\langle \boldsymbol{x}_0(t_f)^2(1-S_{\rm int}[\boldsymbol{x},\boldsymbol{p}])\rangle_0}{\langle 1-S_{\rm int}[\boldsymbol{x},\boldsymbol{p}]\rangle_0}. \tag{34}$$

The interaction action is linear in $\boldsymbol{F}$ and $\boldsymbol{G}$ and so are the averages in (34); these functions are the sum over Fourier modes of functions $\boldsymbol{F}_{\boldsymbol{k}}$ and $\boldsymbol{G}_{\boldsymbol{k}}$, so we can carry out the computation with only one Fourier mode and integrate over all modes at the end. We denote by $S_{{\rm int},\boldsymbol{k}}$ the part of the action due to the interaction with the $\boldsymbol{k}$ mode of the field and compute $\langle S_{{\rm int},\boldsymbol{k}}[\boldsymbol{x},\boldsymbol{p}]\rangle_0$ and $\langle \boldsymbol{x}_0(t_f)^2 S_{{\rm int},\boldsymbol{k}}[\boldsymbol{x},\boldsymbol{p}]\rangle_0$.

In what follows we work at fixed wave vector $\boldsymbol{k}$, so $\tilde{\Delta}(\boldsymbol{k})$, $\tilde{K}(\boldsymbol{k})$, and $\tilde{R}(\boldsymbol{k})$ are pure numbers, and to lighten the notation we will write them Δ, K, and R.

Every average we have to compute is made of terms of the form $\langle \prod_{j=1}^n O_j e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\rangle$, where O_j are operators linear in $\boldsymbol{x}$ and $\boldsymbol{p}$. We will need the following formula, which is easy to derive from Wick's theorem,

$$\left\langle \prod_{j=1}^n O_j e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right\rangle = e^{-\frac{1}{2}\boldsymbol{k}^T\langle \boldsymbol{x}\boldsymbol{x}^T\rangle\boldsymbol{k}} \sum_{J\subset N} i^{|J|}\left(\prod_{j\in J}\boldsymbol{k}\cdot\langle O_j\boldsymbol{x}\rangle\left\langle\prod_{j\notin J}O_j\right\rangle\right), \tag{35}$$

where N is the set $\{1,\ldots,n\}$ and the sum over J denotes the sum over all subsets of N.

We start with $\langle S_{{\rm int},\boldsymbol{k}}[\boldsymbol{x},\boldsymbol{p}]\rangle_0$. This average contains two integrated terms. The first is $\langle \boldsymbol{p}(t)\cdot\boldsymbol{F}_{\boldsymbol{k}}[\boldsymbol{x}(t)-\boldsymbol{x}(s),t-s]\rangle_0$ with $t>s$, which involves

$$\begin{aligned}&\langle \boldsymbol{p}(t)e^{i\boldsymbol{k}\cdot[\boldsymbol{x}(t)-\boldsymbol{x}(s)]}\rangle_0\\ &\quad = i\langle \boldsymbol{p}(t)[\boldsymbol{x}(t)-\boldsymbol{x}(s)]^T\rangle_0\boldsymbol{k}e^{-k^2D_x(t-s)}=0,\end{aligned} \tag{36}$$

with the notation $k^2=|\boldsymbol{k}|^2$, and we have used that $\langle \boldsymbol{p}(t)\boldsymbol{x}(t)^T\rangle_0=0$ because we use the Itô convention in our path integral. For the same reason, $\langle \boldsymbol{p}(t)^T\boldsymbol{G}_{\boldsymbol{k}}(\boldsymbol{x}(t)-\boldsymbol{x}(s),t-s)\boldsymbol{p}(s)\rangle_0=0$. Hence

$$\langle S_{{\rm int},\boldsymbol{k}}[\boldsymbol{x},\boldsymbol{p}]\rangle_0=0. \tag{37}$$

Now we turn to $\langle \boldsymbol{x}_0(t_f)^2 S_{{\rm int},\boldsymbol{k}}[\boldsymbol{x},\boldsymbol{p}]\rangle_0$. We have to compute $\langle \boldsymbol{x}_0(t_f)^2\boldsymbol{p}(t)e^{i\boldsymbol{k}\cdot[\boldsymbol{x}(t)-\boldsymbol{x}(s)]}\rangle_0$ and $\langle \boldsymbol{x}_0(t_f)^2\boldsymbol{p}(t)\boldsymbol{p}(s)^T e^{i\boldsymbol{k}\cdot[\boldsymbol{x}(t)-\boldsymbol{x}(s)]}\rangle_0$, with $s\leqslant t$. The only nonzero term in the first average is

$$\begin{aligned}&\langle \boldsymbol{x}_0(t_f)^2\boldsymbol{p}(t)e^{i\boldsymbol{k}\cdot[\boldsymbol{x}(t)-\boldsymbol{x}(s)]}\rangle_0\\ &\quad=2i\langle \boldsymbol{p}(t)\boldsymbol{x}_0(t_f)^T\rangle_0\langle \boldsymbol{x}_0(t_f)[\boldsymbol{x}(t)-\boldsymbol{x}(s)]^T\rangle_0\boldsymbol{k}e^{-k^2D_x(t-s)}\\ &\quad=-4D_{\boldsymbol{x}}L([0,t_f]\cap[s,t])\boldsymbol{k}e^{-k^2D_x(t-s)}\chi_{[0,t_f]}(t)\\ &\quad=-4D_{\boldsymbol{x}}[t-\max(s,0)]\boldsymbol{k}\,e^{-k^2D_x(t-s)}\chi_{[0,t_f]}(t).\end{aligned} \tag{38}$$

The second contains two nonzero terms, and we get

$$\begin{aligned}&\langle \boldsymbol{x}_0(t_f)^2\boldsymbol{p}(t)\boldsymbol{p}(s)^T e^{i\boldsymbol{k}\cdot[\boldsymbol{x}(t)-\boldsymbol{x}(s)]}\rangle_0\\ &\quad=\{4D_{\boldsymbol{x}}[t-\max(s,0)]\boldsymbol{k}\,\boldsymbol{k}^T-2\chi_{[0,t_f]}(s)\mathbf{1}\}\\ &\qquad\times e^{-k^2D_x(t-s)}\chi_{[0,t_f]}(t).\end{aligned} \tag{39}$$

Now we just have to integrate the above results, and since we are interested in the long-time behavior, we can neglect the terms in $o(t_f)$:

$$\begin{aligned}&\left\langle i\boldsymbol{x}_0(t_f)^2\int \boldsymbol{p}(t)\cdot\boldsymbol{F}_{\boldsymbol{k}}[\boldsymbol{x}(t)-\boldsymbol{x}(s),t-s]\theta(t-s)\,dt\,ds\right\rangle_0\\ &\quad=\frac{4\zeta h^2\kappa_{\boldsymbol{x}}\kappa_\phi D_{\boldsymbol{x}}RK^2k^2}{(D_{\boldsymbol{x}}k^2+\kappa_\phi R\Delta)^2}t_f\end{aligned} \tag{40}$$

and

$$\begin{aligned}&\left\langle \frac{T_\phi}{2}\boldsymbol{x}_0(t_f)^2\int \boldsymbol{p}^T(t)\boldsymbol{G}_{\boldsymbol{k}}[\boldsymbol{x}(t)-\boldsymbol{x}(s),t-s]\boldsymbol{p}(s)\,dt\,ds\right\rangle_0\\ &\quad=2T_\phi h^2\kappa_{\boldsymbol{x}}^2\Delta^{-1}K^2\frac{D_{\boldsymbol{x}}k^4-\kappa_\phi R\Delta k^2}{(D_{\boldsymbol{x}}k^2+\kappa_\phi R\Delta)^2}t_f.\end{aligned} \tag{41}$$

Thus, gathering these two results in (34) gives

$$\langle \boldsymbol{x}_0(t_f)^2\rangle \simeq 2dt_f\left[D_{\boldsymbol{x}}-h^2\kappa_{\boldsymbol{x}}k^2K^2\frac{\kappa_{\boldsymbol{x}}T_\phi D_{\boldsymbol{x}}k^2+(2\zeta D_{\boldsymbol{x}}-\kappa_{\boldsymbol{x}}T_\phi)\kappa_\phi R\Delta}{d\Delta(D_{\boldsymbol{x}}k^2+\kappa_\phi R\Delta)^2}\right]. \tag{42}$$

Integrating over the modes, we get for the effective diffusion constant

$$D_e=D_{\boldsymbol{x}}-\frac{h^2}{d}\int\frac{d^d\boldsymbol{k}}{(2\pi)^d}\kappa_{\boldsymbol{x}}k^2\tilde{K}(\boldsymbol{k})^2\frac{\kappa_{\boldsymbol{x}}T_\phi D_{\boldsymbol{x}}k^2+(2\zeta D_{\boldsymbol{x}}-\kappa_{\boldsymbol{x}}T_\phi)\kappa_\phi\tilde{R}(\boldsymbol{k})\tilde{\Delta}(\boldsymbol{k})}{\tilde{\Delta}(\boldsymbol{k})[D_{\boldsymbol{x}}k^2+\kappa_\phi\tilde{R}(\boldsymbol{k})\tilde{\Delta}(\boldsymbol{k})]^2}. \tag{43}$$

This expression is our main result. The large number of parameters makes its interpretation quite difficult, so we will apply it to some special cases to show the great variety of phenomena that could be described.

B. Infrared behavior and the possibility of anomalous diffusion

The integrals appearing in Eq. (43) will in some cases need to be regularized by either introducing a large k (ultraviolet) cutoff or a small k (infrared) cutoff. The former cutoff corresponds to the existence of a molecular length scale below which the field ϕ cannot fluctuate. The latter case is regularized by a length scale corresponding to the size of the system. The dependence of the diffusion constant on the system size is a sign of the onset of anomalous diffusion as opposed to normal diffusion. To analyze when interaction with the field maintains normal diffusion, we need to analyze the problem in terms of the large distance behavior of the problem. Physically a number of mechanisms can lead to anomalous diffusion [1,41,42], and in potential driven systems subdiffusion can be induced by a divergence in the average time to cross energy barriers. While subdiffusion can be induced by diverging time scales, it can also be induced by the presence of long-range correlations or diverging length scales. While in potential problems trapping in local minima can lead to subdiffusion, long-range correlations in incompressible fields can lead to superdiffusion.

We proceed as in Refs. [20,21] by defining the following exponents associated with the small k behavior of the operators appearing in the problem:

$$\tilde{\Delta}(k) \sim k^{\delta}, \tag{44}$$

$$\tilde{K}(k) \sim k^{\alpha} \tag{45}$$

$$\tilde{R}(k) \sim k^{\rho}. \tag{46}$$

Now, from Eq. (43) we see that there are two distinct regimes that control the small k behavior of the integrals appearing in it. The regimes are $\rho + \delta < 2$ and $\rho + \delta > 2$, and physically we can understand this difference between these regimes. It suffices to note that the free field has a time-dependent length scale $l_\phi(t) \sim t^{\frac{1}{\rho+\delta}}$, whereas the bare diffusion has a length scale $l_{\boldsymbol{x}}(t) \sim t^{\frac{1}{2}}$. The regime where $\rho + \delta < 2$ corresponds to the case where $l_{\boldsymbol{x}}(t) < l_\phi(t)$ and vice versa.

The first, which we refer to as the adiabatic regime, is where $\rho + \delta < 2$. This means that the integrals are dominated at low k by the terms $\tilde{R}\tilde{\Delta}$ rather than the terms $D_{\boldsymbol{x}}k^2$, which means that we effectively find ourselves in the adiabatic limit where $\kappa_{\boldsymbol{x}} \ll \kappa_\phi$ as defined in Refs. [19–21], which physically corresponds to field dynamics that is much quicker than the bare diffusion of the tracer. In this case the field is in a local equilibrium about the tracer, a fact that is seen in the adiabatic path-integral approach of Ref. [19]. In this case there is a critical dimension d_c given by

$$d_c = \rho - 2\alpha + 2\delta - 2 \tag{47}$$

such that for $d > d_c$ the diffusion is normal. For $d < d_c$ the diffusion will be anormal. There is a second possibility where $\rho + \delta > 2$. In this case the bare diffusion of the particle is more rapid than the field dynamics, and we are in the opposite limit to the adiabatic one. Here we find that

$$d_c = \delta - 2\alpha. \tag{48}$$

C. Application to some special cases

1. Stochastic dynamics with detailed balance

We first analyze the case where the particle and the field see the same temperature, $T_{\boldsymbol{x}} = T_\phi = T$, and the dynamics obeys detailed balance, i.e., $\zeta = 1$. The effective diffusion constant is thus

$$D_e^{\text{db}} = D_{\boldsymbol{x}} \left\{ 1 - \frac{h^2}{d} \int \frac{d^d\boldsymbol{k}}{(2\pi)^d} \frac{\kappa_{\boldsymbol{x}} k^2 \tilde{K}(\boldsymbol{k})^2}{\tilde{\Delta}(\boldsymbol{k})[D_{\boldsymbol{x}}k^2 + \kappa_\phi \tilde{R}(\boldsymbol{k})\tilde{\Delta}(\boldsymbol{k})]} \right\}. \tag{49}$$

This result is exactly what was found in Ref. [22] via a Kubo formula formalism that we emphasize applies only to this particular case. By inspecting Eq. (49) we see that the correction to the bare diffusion constant is always negative, and the diffusion is thus slowed by its coupling to the field. The fact that the diffusion is slowed for all values of h and not just in the regime of small h can be shown explicitly within the Kubo formalism [22]. A physical explanation for the slowing of diffusion in this case can be found in studies of the drag on a particle coupled to a field. The reaction of the field to the particle is to create a polaron-like deformation of the field about the particle; however, a moving particle has a polaron that is not symmetric with respect to the front and rear of the particle. This deformation generates a drag force that tends to pull the particle backward [20,21]. The above result also agrees, to $O(h^2)$ in perturbation theory, with studies of the drag on proteins in membranes that couple to membrane curvature [13,19,22] in the adiabaitc limit defined above. These studies analyzed the adiabatic limit via, respectively, phenomonological arguments, a saddle-point approximation in the path-integral formulation, and an operator formalism. The former two studies analyzed the case of quadratic couplings, but in the limits where the coupling becomes linear the results agree with those obtained here and in Ref. [22]. These results are obtained from Eq. (49) by taking the *adiabatic* limit $D_{\boldsymbol{x}} \to 0$ in the denominator of the integral.

2. Passive diffusion

For passive diffusion, that is, $\zeta = 0$, and still with equal temperatures for the particle and field, we have

$$D_e^{\text{pass}} = D_{\boldsymbol{x}} \left\{ 1 - \frac{h^2}{d} \int \frac{d^d\boldsymbol{k}}{(2\pi)^d} \kappa_{\boldsymbol{x}} k^2 \tilde{K}(\boldsymbol{k})^2 \times \frac{D_{\boldsymbol{x}}k^2 - \kappa_\phi \tilde{R}(\boldsymbol{k})\tilde{\Delta}(\boldsymbol{k})}{\tilde{\Delta}(\boldsymbol{k})[D_{\boldsymbol{x}}k^2 + \kappa_\phi \tilde{R}(\boldsymbol{k})\tilde{\Delta}(\boldsymbol{k})]^2} \right\}. \tag{50}$$

This result shows that, depending on the speeds of evolution $\kappa_{\boldsymbol{x}}$ and κ_ϕ of the particle and the field, the particle's diffusion may increase or decrease. This is in agreement with the following intuition: With a slow field we are close to diffusion in a quenched potential, which slows the diffusion due to trapping in local minima that are temporally persistent. However, when the field fluctuates quickly, the field *fluctuations kick the particle along,* thus adding to the effective random force the particle experiences. However, this picture is not totally valid: There appears an optimal value of κ_ϕ at which the perturbative enhancement of the tracer's diffusion constant is maximal. If the field fluctuates too quickly, the effect of field fluctuations simply averages out to zero. This result is quite difficult to understand physically, but it resembles closely the phenomenon of stochastic resonance [29–31], where the application of a periodic but deterministic potential to Brownian particles can show an optimal frequency at which the particle dispersion is maximized. Here the optimal value of κ_ϕ depends, in particular, on the bare diffusivity $\kappa_{\boldsymbol{x}}$ of the particle; thus two different species will react differently to the same external field. We note that this type of phenomenon can be used to sort molecules [32].

3. Particle not connected to a thermal bath

Another case which could have physical relevance is where the particle is not connected to a thermal bath, i.e., $T_{\boldsymbol{x}} = 0$, and energy enters the system only via the fluctuations of the field. Hence in the absence of coupling to the field, the particle cannot diffuse, as $D_{\boldsymbol{x}} = 0$. Our result shows that the fluctuations of the field will induce a nonzero diffusion constant for the particle:

$$D_e^{T_x=0} = T_\phi \frac{h^2}{d} \int \frac{d^d \boldsymbol{k}}{(2\pi)^d} \frac{\kappa_x^2 k^2 \tilde{K}(\boldsymbol{k})^2}{\kappa_\phi \tilde{R}(\boldsymbol{k}) \tilde{\Delta}(\boldsymbol{k})^2}. \quad (51)$$

The effective diffusion constant is now proportional to the temperature seen by the field T_ϕ: In some sense, the field acts as a thermal bath for the particle. Note that the effect of field fluctuations is to speed up the diffusion from a zero diffusion constant to that given by Eq. (51); this is in agreement with the physical intuition that the field fluctuations will help the particle to disperse. Interestingly, we see that when $D_x = 0$, the result for D_e is independent of ζ, and the active and passive cases have the same diffusion constant.

V. NUMERICAL SIMULATIONS

In this section we test out our theoretical predictions against the numerical simulations of a toy model in one dimension. We should bear in mind that the simulation of the diffusion of active tracers is more computationally intensive than that for passive tracers. In the latter case we can simulate the diffusion of an ensemble of independent tracers on a given dynamical realization of the fluctuating field. In the former case, however, we must follow the diffusion of a single particle in the fluctuating field as for a system with more than one particle the coupling to the field introduces interactions between the particles [23,24]. In what follows we will numerically compute the evolution of Eq. (5) in Fourier space and while integrating Eq. (2) in real space to generate single-particle trajectories, which are then ensemble averaged to estimate the diffusion constant.

A. Numerical model

We consider the simplest model for our numerical simulations, $d = 1$, and we take a finite number of modes, with $k = \pm n$, $1 \leqslant n \leqslant N$. We also take $\tilde{\Delta}(k) = k^2$, $\tilde{K}(k) = 1$, and $\tilde{R}(k) = 1$. We set $T_\phi = 1$, $\kappa_x = 1$. For this choice of parameters, our result [Eq. (43)] reads

$$D_e = D_x - h^2 \frac{D_x + \kappa_\phi(2\zeta D_x - 1)}{\pi (D_x + \kappa_\phi)^2} \sum_{n=1}^{N} \frac{1}{n^2}. \quad (52)$$

We will perform four simulations: First, we consider stochastic dynamics with detailed balance, and let the coupling h vary, to explore the range of validity of our perturbative result. Then we will simulate the three special cases described above, for different values of the speed of evolution of the field κ_ϕ. Each simulation is performed for one mode and 10 modes.

In the simulations, we let one particle evolve in the field for a long time $\tau \gg \kappa_x^{-1}, \kappa_\phi^{-1}$, and we measure its position at a fixed set of times. We repeat this simulation a large number of times (around 10^5) and, using these measurements, we compute $\langle \boldsymbol{x}(t)^2 \rangle / 2dt$, where t is the measurement time. For large t, this function fluctuates around a mean value, which gives us the effective diffusion constant. When these fluctuations are not small, they are taken into account with error bars on the plots.

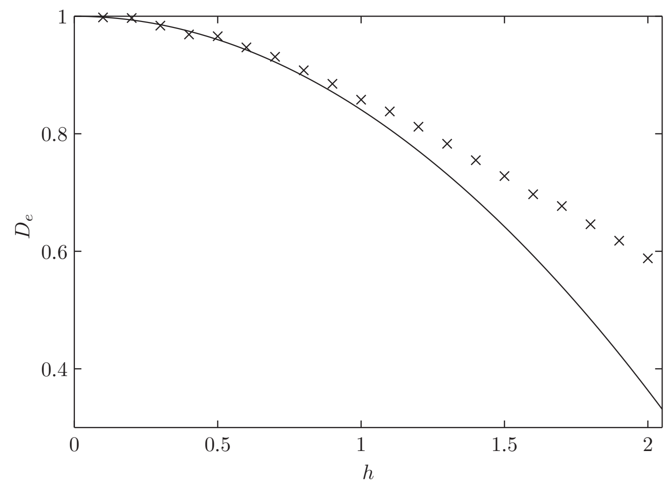

FIG. 1. Effective diffusion coefficient for stochastic dynamics with detailed balance (Gaussian ferromagnet with model A dynamics: $\tilde{\Delta} = k^2$, $\tilde{R} = 1$, $\tilde{K} = 1$) with a single-mode field as a function of the coupling constant h. The crosses represent numerical simulations; the solid line is the perturbative result Eq. (52).

B. Validity range of the perturbative result

The first question that arises is: To what extent is our perturbative result valid? To find the validity range for h, we take stochastic dynamics with detailed balance, with one mode and $\kappa_\phi = 1$, and look at the effective diffusion coefficient as a function of h. The comparison between the simulations and the result (52) is given Fig. 1, and it shows that our computation is valid (i.e., the relative error is less than 5%) for $h \lesssim 1.2$. A more relevant criterion is, however, to what extent the deviation from the bare result can be predicted by our result. Figure 1 shows that the theory predicts the deviations of the diffusion constant from its bare value in the region where the diffusion constant deviates of the order of 15% –20% from its bare value.

C. Stochastic dynamics with detailed balance

We can also vary the rate of evolution of the field κ_ϕ. We set $D_x = 1$ and plot $D_e^{\text{db}}(\kappa_\phi)$ for one mode and $h = 1$, and for 10 modes and $h = 0.5$, comparing the numerical simulations results with (52). The results are shown in Fig. 2. For one mode and $h = 1$, at the border of the range of validity of the perturbative approach, our results are in quite good agreement with the simulations.

D. Passive diffusion

For passive diffusion, the results of numerical simulations are shown Fig. 3 and compared to Eq. (52). We see that the analytical predictions are in good agreement with the results of simulations. In particular we see that depending on the relative values of κ_x and κ_ϕ, the diffusion is either slowed down or speeded up. For small values of κ_ϕ the diffusion is reduced; however, on increasing κ_ϕ the diffusion speeds up and passes through a maximum before decaying

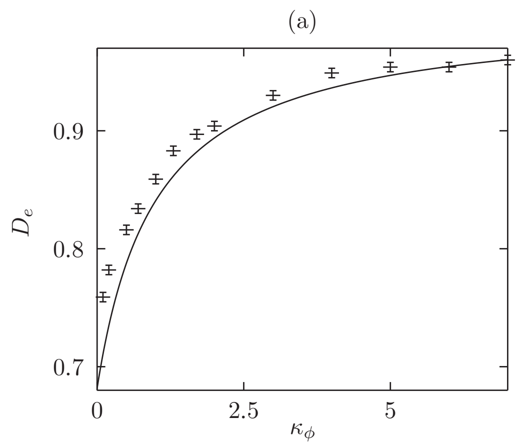

FIG. 2. Effective diffusion coefficient for stochastic dynamics with detailed balance (Gaussian ferromagnet with model A dynamics: $\tilde{\Delta} = k^2$, $\tilde{R} = 1$, $\tilde{K} = 1$) as a function of κ_ϕ with $D_{\boldsymbol{x}} = 1$ and (a) 1 mode and $h = 1$ (b) 10 modes and $h = 0.5$: numerical simulations (crosses) and perturbative results (solid lines).

toward the bare value $D_{\boldsymbol{x}} = 1$ as predicted by our perturbative calculations.

E. Particle not connected to any thermal bath

The results for numerical simulations when the particle is not connected to any thermal bath are shown Fig. 4. They are in good agreement with Eq. (52) for $\kappa_\phi \gtrsim 1.5$. When $\kappa_\phi \to 0$, according to our computation, the effective diffusion coefficient diverges, whereas physically it should go to zero, which is confirmed by the simulations. This discrepancy comes from the fact that we neglected the terms in $o(t_f)$ in our computation of $\langle \boldsymbol{x}(t_f)^2 S_{\text{int}} \rangle_0$, and we took the limit $t_f \to \infty$ before taking the limit $\kappa_\phi \to 0$.

VI. DIFFUSION COEFFICIENT FROM THE PROBABILITY DENSITY FOR THE PASSIVE CASE

Here we use a perturbation expansion of the Fokker-Planck equation in order to compute the perturbative correction for passive diffusion in a fluctuating field. The basic formalism is described in Refs. [1,2]. In terms of the general model of this paper we are thus considering the special case $\zeta = 0$ and $T_{\boldsymbol{x}} = T_\phi = T$.

First, we give the elementary equations for the pure Brownian motion. The probability density function $P_0(\boldsymbol{x},t)$ of a particle starting at $\boldsymbol{x} = \boldsymbol{0}$ when $t = 0$, setting $P_0(\boldsymbol{x},t < 0) = 0$, satisfies

$$\dot{P}_0(\boldsymbol{x},t) = \boldsymbol{\nabla} \cdot [D_{\boldsymbol{x}} \boldsymbol{\nabla} P(\boldsymbol{x},t)] + \delta(\boldsymbol{x})\delta(t). \tag{53}$$

We can write this equation in terms the free diffusion operator $H_0 = \partial_t - \boldsymbol{\nabla} \cdot (D_{\boldsymbol{x}} \boldsymbol{\nabla})$:

$$H_0 P_0(\boldsymbol{x},t) = \delta(\boldsymbol{x})\delta(t), \tag{54}$$

and Fourier transform $\tilde{P}_0(\boldsymbol{k},\omega) = \int d\boldsymbol{x}\; dt\, P_0(\boldsymbol{x},t) e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t)}$ is given by

$$\tilde{P}_0(\boldsymbol{k},\omega) = \frac{1}{D_{\boldsymbol{x}} k^2 + i\omega}. \tag{55}$$

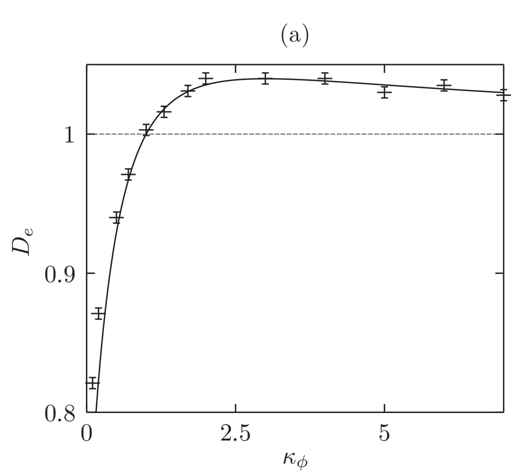

FIG. 3. Effective diffusion coefficient for passive diffusion (Gaussian ferromagnet with model A dynamics: $\tilde{\Delta} = k^2$, $\tilde{R} = 1$, $\tilde{K} = 1$) as a function of κ_ϕ with $D_{\boldsymbol{x}} = 1$ and (a) 1 mode and $h = 1$ (b) 10 modes and $h = 0.5$: numerical simulations (crosses) and perturbative results (solid lines).

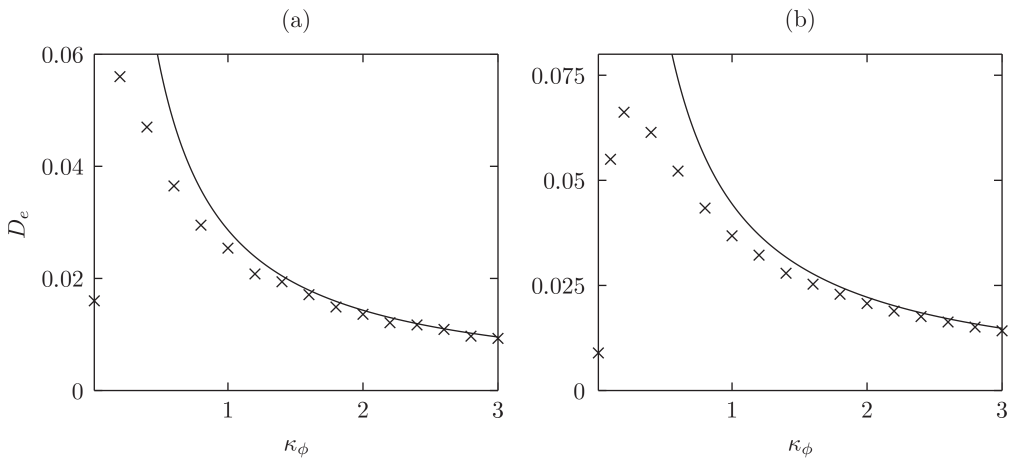

FIG. 4. Effective diffusion coefficient for a particle not connected to any thermal bath (Gaussian ferromagnet with model A dynamics: $\tilde{\Delta} = k^2$, $\tilde{R} = 1$, $\tilde{K} = 1$) as a function of κ_ϕ with $\kappa_{\boldsymbol{x}} = 1$ and (a) 1 mode and $h = 0.3$, (b) 10 modes and $h = 0.3$: numerical simulations (crosses) and perturbative results (solid lines).

The effective diffusion coefficient D_e of a process can thus be extracted from the Fourier transform of its probability density function $\tilde{P}(\boldsymbol{k},\omega)$ via [1,2]

$$D = \left[\lim_{|\boldsymbol{k}|\to 0} k^2 \tilde{P}(\boldsymbol{k},0)\right]^{-1}. \tag{56}$$

We are indeed interested in the large distance behavior of our system, which is why the effective diffusion coefficient is given by the small wave-vector behavior. With this equation in hand, our strategy is simple: Compute the probability density function for the passive diffusion and extract the effective diffusion coefficient.

Now, we introduce a given field $\phi(\boldsymbol{x},t)$. For the general model described above, Eq. (2) gives the diffusion operator, which replaces H_0 in Eq. (54):

$$H = H_0 + H_{\text{int}} = \partial_t - \nabla \cdot \{D_{\boldsymbol{x}}\nabla + h\kappa_{\boldsymbol{x}}[\nabla(K\phi)(\boldsymbol{x},t)]\}. \tag{57}$$

The equation satisfied by the Fourier transform $\tilde{P}(\boldsymbol{k},\omega)$ is

$$\widetilde{HP}(\boldsymbol{k},\omega) = (i\omega + D_{\boldsymbol{x}}k^2)\tilde{P}(\boldsymbol{k},\omega) + h\kappa_{\boldsymbol{x}} \int \frac{d\boldsymbol{q}\,d\nu}{(2\pi)^{d+1}} \boldsymbol{k}\cdot\boldsymbol{q}\tilde{K}(\boldsymbol{q})\tilde{\phi}(\boldsymbol{q},\nu) \times \tilde{P}(\boldsymbol{k}-\boldsymbol{q},\omega-\nu) = 1, \tag{58}$$

so $\tilde{P}(\boldsymbol{k},\omega)$ is given by the integral equation:

$$\tilde{P}(\boldsymbol{k},\omega) = \tilde{P}_0(\boldsymbol{k},\omega)\left[1 - h\kappa_{\boldsymbol{x}} \int \frac{d\boldsymbol{q}\,d\nu}{(2\pi)^{d+1}} \boldsymbol{k}\cdot\boldsymbol{q} \times \tilde{K}(\boldsymbol{q})\tilde{\phi}(\boldsymbol{q},\nu)\tilde{P}(\boldsymbol{k}-\boldsymbol{q},\omega-\nu)\right]. \tag{59}$$

In this equation, the probability density function is that of pure Brownian motion, perturbed to the order h. Iterating this equation gives an explicit expression of $\tilde{P}(\boldsymbol{k},\omega)$ up to the desired order of h.

Once we have the equation for $\tilde{P}(\boldsymbol{k},\omega)$ for a given field we need to extract the effective diffusion coefficient and proceed by averaging (59) over the configurations of the field $\phi(\boldsymbol{x},t)$ (which does not depend on the particle position). The field has a Gaussian probability density function with a two-point correlation function that can easily be computed from Eq. (9):

$$\langle\tilde{\phi}(\boldsymbol{q},\nu)\tilde{\phi}(\boldsymbol{q}',\nu')\rangle = \frac{2T\kappa_\phi\tilde{R}(\boldsymbol{q})}{\nu^2 + [\kappa_\phi\tilde{R}(\boldsymbol{q})\tilde{\Delta}(\boldsymbol{q})]^2}(2\pi)^{d+1}\delta(\boldsymbol{q}+\boldsymbol{q}')\delta(\nu+\nu'). \tag{60}$$

Now we have to insert this average in an explicit perturbative expansion of $\tilde{P}(\boldsymbol{k},\omega)$ given by (59). The lowest nonzero order is the second order: The field $\phi(\boldsymbol{x},t)$ has to appear at least twice. Moreover, we will obtain the probability density function to the order h^2, which is exactly what we did with the path-integral method. Computing $\tilde{P}(\boldsymbol{k},\omega)$ to the order h^2 and averaging the field out leads to

$$\langle\tilde{P}(\boldsymbol{k},\omega)\rangle = \tilde{P}_0(\boldsymbol{k},\omega)\left[1 - 2h^2\tilde{P}_0(\boldsymbol{k},\omega)T\kappa_\phi\kappa_{\boldsymbol{x}}^2 \int \frac{d\boldsymbol{q}\,d\nu}{(2\pi)^{d+1}} \frac{\boldsymbol{k}\cdot\boldsymbol{q}\,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{q}\tilde{R}(\boldsymbol{q})\tilde{K}(\boldsymbol{q})^2}{\nu^2 + [\kappa_\phi\tilde{R}(\boldsymbol{q})\tilde{\Delta}(\boldsymbol{q})]^2}\tilde{P}_0(\boldsymbol{k}-\boldsymbol{q},\omega-\nu)\right]. \tag{61}$$

Then we can restrict ourselves to $\omega = 0$, use the expression (55) and integrate over ν, using $\int \frac{d\nu}{2\pi}\frac{1}{(i\nu+\alpha)(\nu^2+\beta^2)} = \frac{1}{2\beta(\alpha+\beta)}$:

$$\langle\tilde{P}(\boldsymbol{k},0)\rangle = \frac{1}{D_{\boldsymbol{x}}k^2}\left[1 - \frac{h^2\kappa_{\boldsymbol{x}}}{k^2}\int \frac{d\boldsymbol{q}}{(2\pi)^d}\frac{\boldsymbol{k}\cdot\boldsymbol{q}\,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{q}\tilde{K}(\boldsymbol{q})^2}{\tilde{\Delta}(\boldsymbol{q})[D_{\boldsymbol{x}}(\boldsymbol{k}-\boldsymbol{q})^2 + \kappa_\phi\tilde{R}(\boldsymbol{q})\tilde{\Delta}(\boldsymbol{q})]}\right]. \tag{62}$$

Finally, we just need to determine to behavior of the above expression when $|\boldsymbol{k}| \to 0$; a straightforward computation gives

$$\langle \tilde{P}(\boldsymbol{k},0)\rangle \underset{|\boldsymbol{K}|\to 0}{\sim} \frac{1}{D_x k^2}\left[1 + \frac{h^2\kappa_x}{d}\int \frac{d\boldsymbol{q}}{(2\pi)^d}\frac{q^2\tilde{K}(\boldsymbol{q})^2[D_x q^2 - \kappa_\phi \tilde{R}(\boldsymbol{q})\tilde{\Delta}(\boldsymbol{q})]}{\tilde{\Delta}(\boldsymbol{q})[D_x q^2 + \kappa_\phi \tilde{R}(\boldsymbol{q})\tilde{\Delta}(\boldsymbol{q})]^2}\right], \tag{63}$$

and we recover the effective diffusion coefficient given in (50):

$$D_e^{\text{pass}} = D_x\left\{1 - \frac{h^2\kappa_x}{d}\int \frac{d^d\boldsymbol{k}}{(2\pi)^d} q^2\tilde{K}(\boldsymbol{q})^2 \frac{D_x q^2 - \kappa_\phi \tilde{R}(\boldsymbol{q})\tilde{\Delta}(\boldsymbol{q})}{\tilde{\Delta}(\boldsymbol{q})[D_x q^2 + \kappa_\phi \tilde{R}(\boldsymbol{q})\tilde{\Delta}(\boldsymbol{q})]^2}\right\}. \tag{64}$$

VII. CONCLUSIONS

We have analyzed the diffusive behavior of a tracer particle diffusing in a time-dependent Gaussian potential in the limit of weak coupling between the particle and the field. The method has the advantage that it can be applied to a wide range of models and not just the cases of passive diffusion or active diffusion with detailed balance. In these two aforementioned cases the method agrees with results obtained, respectively, via a perturbation expansion of the Fokker Planck equation and a Kubo formulation. We have also been able to look at nonequilibrium systems with variable feedback of the tracer on the field and systems where the field and the tracer are subject to thermal noise of different temperatures. The range of behavior seen in the late-time diffusion coefficient is quite rich, and depending on the models considered, coupling to the field can either slow down or speed up the diffusion. The speeding up or slowing down of diffusion and the possibility of a form of stochastic resonance depends on the relative rates of the dynamics of the fluctuating field and the bare diffusion constant of the tracer. Extensions of the work done here beyond the weak coupling approximation would be interesting to pursue; it is perhaps possible to apply Gaussian or mode-coupling-type approximations [7] to analyze the regime of strong interaction and perhaps even explore whether field fluctuations can lead to anomalous diffusion. In addition it would be interesting to see how the effects found here are modified when the coupling between the field and the tracer are nonlinear, for instance, quadratic. Such couplings are natural in systems where the tracer does not break the symmetry of the fluctuating field but rather enhances or suppresses its fluctuations. An example is a stiff membrane insertion that suppresses fluctuations in membrane curvature [13]. A final point that would be interesting to address is what is the effect of a finite density of tracers for the active system under stochastic dynamics obeying detailed balance. As mentioned previously there will be induced interactions between the particles [23,24], and it would be interesting to see how this modifies the effective diffusion constant.

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