École Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Exercises

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1 Gradient dynamics for the anisotropic RGL equation

The two-dimensional anisotropic amplitude equation for stripes emerging from an isotropic I-s instability is

$$\partial_t A = A + \left(\partial_x - \frac{\mathrm{i}}{2}\partial_{yy}\right)^2 A - |A|^2 A, \tag{1}$$

where $A(x,y,t)$ is a complex field.

1. Write this equation in a gradient form with a functional $\mathcal{F}$.

2. Is $\mathcal{F}$ a Lyapunov functional?

2 Multi-scale expansion of model A

We consider model A for a real field $u(\boldsymbol{r},t)$:

$$\partial_t u = -\Gamma(u)\frac{\delta\mathcal{F}}{\delta u}, \tag{2}$$

where $\Gamma(u) > 0$ and the free energy $\mathcal{F}$ reads

$$\mathcal{F}[u] = \int \left(\frac{\beta}{2}[\nabla u(\boldsymbol{r})]^2 + v(u(\boldsymbol{r}))\right)\mathrm{d}\boldsymbol{r}, \tag{3}$$

where $\beta > 0$. We assume that v is a positive and even function of u.

3. Study the linear stability of the state $u = 0$.

We write $v''(0) = -\epsilon a$ where $a > 0$ and $\epsilon > 0$ is a small dimensionless parameter.

4. How do the spatial and temporal scales behave when ϵ is small?

5. Perform the multi-scale expansion around $u = 0$ with the small parameter ϵ.

3 Blow-up in the Fisher-Kolmogorov equation

We consider the Fisher-Kolmogorov equation equation for a real field $u(\boldsymbol{r},t)$ over a domain Ω with volume V, with a Neumann boundary condition $\hat{\boldsymbol{n}}(\boldsymbol{r})\cdot\nabla u(\boldsymbol{r},t) = 0$ for $\boldsymbol{r}\in\partial\Omega$, where $\hat{\boldsymbol{n}}(\boldsymbol{r})$ is a unit vector normal to the boundary:

$$\partial_t u = u + u^2 + \nabla^2 u. \tag{4}$$

We define the mass

$$m(t) = \frac{1}{V}\int_\Omega u(\boldsymbol{r},t)\mathrm{d}\boldsymbol{r}. \tag{5}$$

6. Show that the mass obeys $\dot{m}(t) \geq m(t) + m(t)^2$.

7. Show that if $m(0) > 0$, then

$$m(t) \geq \frac{1}{\mathrm{e}^{t_0 - t} - 1}, \tag{6}$$

where t_0 is a constant to determine. Discuss the behavior of the system.