Ecole Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

One-sided traffic model and the Burgers equation

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1 One-sided traffic model and the Burgers equation

We consider a simple model for single-lane traffic flow, where cars have a position $X_n(t)$ with n an integer, and $X_{n+1}(t) \geq X_n(t)$. We assume that the drivers watch only in the direction $\hat{\boldsymbol{e}}_x$, corresponding to the front of their vehicles. We assume that they only see their first car in their front, so that their velocity depend only on the distance to this car:

$$\dot{X}_n(t) = V(\ell_n(t)) \tag{1}$$

$$\ell_n(t) = X_{n+1}(t) - X_n(t). \tag{2}$$

This model is thus entirely determined by the velocity function $V(\ell)$. In this tutorial, we will first determine the condition for a homogeneous flow to be linearly stable. Then, we will take the hydrodynamic limit and derive an equation for the car density $\rho(x,t)$. Last, we will show that small density fluctuations obey the Burgers equation.

1.1 Linear stability analysis

1. * Consider a small perturbation from a homogeneous state where all cars are equidistant and determine the condition for the perturbations of this state to be stable.

1.2 Hydrodynamics

We focus on large distances and introduce a continuous car label $y = \epsilon n = \mathcal{O}(1)$, so that the positions of the cars and the distances between adjacent cars are $X(y,t)$ and $\ell(y,t) = X(y+\epsilon,t) - X(y,t)$.

2. Write the evolution equation of the position field $X(y,t)$. Is it local?

3. Taylor expand the distance $\ell(y,t)$ to the order ϵ^2 and use it in the dynamics of the position field to make it local.

4. Explain why we can define the density in the Lagrangian frame as

$$\hat{\rho}(y,t) = \frac{1}{\epsilon \partial_y X(y,t)}. \tag{3}$$

5. * Determine the dynamics of $\hat{\rho}(y,t)$.
The density in the Eulerian frame, $\rho(x,t)$, is defined by $\hat{\rho}(y,t) = \rho(X(y,t),t)$.

6. ** Derive the definition of $\rho(x,t)$ and combine it with the dynamics of $\hat{\rho}(y,t)$ to determine the dynamics of $\rho(x,t)$. In the calculation, you will have to compute $\partial_y f(X(y,t),t)$ for some function $f(x,t)$. Show that $\rho(x,t)$ obeys a conservation equation where the current $j(x,t)$ is given by

$$j = \rho V\left(\frac{1}{\rho}\right) + \frac{1}{2}\partial_x V\left(\frac{1}{\rho}\right). \tag{4}$$

Figure 1: Hydraulic jump in a sink. By User Zeimusu on en.wikipedia - James Kilfiger took this picture, CC BY-SA 3.0.

7. * We assume weak density fluctuations $\phi(x,t) = \rho(x,t) - \bar{\rho} \sim \eta$ and small gradients, $\partial_x \sim \eta$, where $\eta \ll 1$ is a small parameter. Determine the equation for the density fluctuations $\phi(x,t)$ to order η^2.

8. Use a Galilean transform to simplify the equation for $\phi(x,t)$ and find that it follows the Burgers equation.

9. What is the condition for the perturbation $\phi(x,t)$ to be linearly stable? Compare this condition with the one obtained in the first question.

10. Discuss the travelling solutions $\phi(x,t) = f(x - vt)$ that could appear between two homogeneous states, $\lim_{x\to\pm\infty} f(x,t) = \phi_\pm$.

2 Hydraulic jump

Here we apply conservation laws to characterize another type of shock: the hydraulic jump. An hydraulic jump is easily created in a sink: the water flowing down from a tap gives rise to a bulge in the water flow centered around the jet, which separated a central region where the fluid is shallow and fast and an outer region where the fluid is deeper and slow (Fig. 1).

We consider the simple situation where the jump occurs in a channel of constant width and is stationary. We denote $h(x)$ the height of the fluid at the position x along the channel, and $U(x)$ its velocity along the channel. Our goal is to relate the height and velocities (h', U') of the flow after the jump to the ones before the jump, (h, U).

11. What relation imposes mass conservation?

12. * Write down the momentum conservation, neglecting the momentum lost by friction. Use the result and the previous question to relate (h', U') to (h, U).