Ecole Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

One-sided traffic model and the Burgers equation (solution)

Vincent Démery, Olivier Pierre-Louis

1 One-sided traffic model and the Burgers equation

1.1 Linear stability analysis

1. Here we consider a small perturbation from a homogeneous state where all cars are equidistant, with distance $\bar{\ell}$ and we denote

$$X_n(t) = n\bar{\ell} + +V(\bar{\ell})t + U_n(t), \tag{1}$$

where $|U_n(t)| \ll \bar{\ell}$. The evolution of $U_n(t)$ is given by

$$\dot{U}_n(t) = V(\bar{\ell} + U_{n+1}(t) - U_n(t)) - V(\bar{\ell}) \tag{2}$$
$$\simeq V'(\bar{\ell})[U_{n+1}(t) - U_n(t)], \tag{3}$$

to first order in $U_n(t)$.

We define the Fourier transform

$$\tilde{U}(\phi, t) = \sum_n \mathrm{e}^{-\mathrm{i}n\phi} U_n(t). \tag{4}$$

Noting that

$$\sum_n \mathrm{e}^{-\mathrm{i}n\phi} U_{n+1}(t) = \sum_k \mathrm{e}^{-\mathrm{i}(k-1)\phi} U_k(t) = \mathrm{e}^{\mathrm{i}\phi} \tilde{U}(\phi, t), \tag{5}$$

the Fourier transform of the evolution equation above is

$$\partial_t \tilde{U}(\phi, t) = \left[\mathrm{e}^{\mathrm{i}\phi} - 1\right] V'(\bar{\ell}) \tilde{U}(\phi, t). \tag{6}$$

Since $\Re(\mathrm{e}^{\mathrm{i}\phi} - 1) = \cos(\phi) - 1 < 0$, the perturbation is stable if $V'(\bar{\ell}) > 0$.

1.2 Hydrodynamics

We focus on large distances and introduce a continuous car label $y = \epsilon n = \mathcal{O}(1)$, so that the positions of the cars and the distances between adjacent cars are $X(y, t)$ and $\ell(y, t) = X(y + \epsilon, t) - X(y, t)$.

2. The dynamics of the position field $X(y, t)$ is given by

$$\partial_t X(y, t) = V(\ell(y, t)). \tag{7}$$

Because of the definition of $\ell(y, t)$, this equation is non-local in y.

3. To the order ϵ^2, the distance is

$$\ell(y, t) \simeq \epsilon \partial_y X(y, t) + \frac{\epsilon^2}{2} \partial_y^2 X(y, t). \tag{8}$$

Inserting this expansion in the dynamics (7) and Taylor expanding $V(\ell(y, t))$ leads to the local equation

$$\partial_t X(y, t) = V(\epsilon \partial_y X(y, t)) + \frac{\epsilon^2}{2} \partial_y^2 X(y, t) V'(\epsilon \partial_y X(y, t)). \tag{9}$$

4. The density $\hat{\rho}(y,t) = 1/\epsilon\partial_y X(y,t)$ is obtained from the lowest order term of the Taylor expansion of $\ell(y,t)$. The next order contains the asymmetry of the dynamics, which should not be included in the definition of the density.

5. The dynamics of $\hat{\rho}(y,t)$ is obtained by the usual derivation rules:

$$\partial_t\hat{\rho}(y,t) = -\frac{\partial_y\partial_t X(y,t)}{\epsilon[\partial_y X(y,t)]^2} \tag{10}$$

$$= -\epsilon\hat{\rho}(y,t)^2\partial_y\partial_t X(y,t) \tag{11}$$

$$= -\epsilon\hat{\rho}(y,t)^2\partial_y\left[V(\epsilon\partial_y X(y,t)) + \frac{\epsilon^2}{2}\partial_y^2 X(y,t)V'(\epsilon\partial_y X(y,t))\right] \tag{12}$$

$$= -\epsilon\hat{\rho}(y,t)^2\partial_y\left[V\left(\frac{1}{\hat{\rho}(y,t)}\right) + \frac{\epsilon}{2}\partial_y\left(\frac{1}{\hat{\rho}(y,t)}\right)V'\left(\frac{1}{\hat{\rho}(y,t)}\right)\right] \tag{13}$$

$$= -\epsilon\hat{\rho}(y,t)^2\partial_y\left[V\left(\frac{1}{\hat{\rho}(y,t)}\right) - \frac{\epsilon\partial_y\hat{\rho}(y,t)}{2\hat{\rho}(y,t)^2}V'\left(\frac{1}{\hat{\rho}(y,t)}\right)\right]. \tag{14}$$

6. Deriving the definition of $\rho(x,t)$ with respect to time, we find

$$\partial_t\hat{\rho}(y,t) = \partial_t\rho(X(y,t),t) + \partial_t X(y,t)\partial_x\rho(X(y,t),t). \tag{15}$$

Combining with the evolution of $\hat{\rho}(y,t)$, we find

$$\partial_t\rho(X(y,t),t) = -\left[\partial_x\rho(X(y,t),t) + \epsilon\hat{\rho}(y,t)^2\partial_y\right]\left[V\left(\frac{1}{\hat{\rho}(y,t)}\right) - \frac{\epsilon\partial_y\hat{\rho}(y,t)}{2\hat{\rho}(y,t)^2}V'\left(\frac{1}{\hat{\rho}(y,t)}\right)\right]. \tag{16}$$

To express the right-hand-side as a function of $\rho(x,t)$, we note that for some function $f(x,t)$,

$$\partial_y f(X(y,t),t) = \partial_y X(y,t)\partial_x f(X(y,t),t) = \frac{\partial_x f(X(y,t),t)}{\epsilon\rho(X(y,t),t)}. \tag{17}$$

Using this relation in (16), we find

$$\partial_t\rho(X(y,t),t) = -\left[\partial_x\rho(X(y,t),t) + \epsilon\rho(X(y,t),t)^2\partial_y\right]\left[V\left(\frac{1}{\rho(X(y,t),t)}\right) - \frac{\partial_x\rho(X(y,t),t)}{2\rho(X(y,t),t)^3}V'\left(\frac{1}{\rho(X(y,t),t)}\right)\right]. \tag{18}$$

Finally, replacing $X(y,t)$ by x in the expression, we get

$$\partial_t\rho(x,t) = -\left[\partial_x\rho(x,t) + \rho(x,t)\partial_x\right]\left[V\left(\frac{1}{\rho(x,t)}\right) - \frac{\partial_x\rho(x,t)}{2\rho(x,t)^3}V'\left(\frac{1}{\rho(x,t)}\right)\right] \tag{19}$$

$$= -\partial_x\left(\rho(x,t)\left[V\left(\frac{1}{\rho(x,t)}\right) - \frac{\partial_x\rho(x,t)}{2\rho(x,t)^3}V'\left(\frac{1}{\rho(x,t)}\right)\right]\right) \tag{20}$$

$$= -\partial_x\left[\rho(x,t)V\left(\frac{1}{\rho(x,t)}\right) - \frac{\partial_x\rho(x,t)}{2\rho(x,t)^2}V'\left(\frac{1}{\rho(x,t)}\right)\right]. \tag{21}$$

To simplify further, we note that (discarding the arguments of $\rho(x,t)$),

$$\partial_x V\left(\frac{1}{\rho}\right) = -\frac{\partial_x\rho}{\rho^2}V'\left(\frac{1}{\rho}\right), \tag{22}$$

so that we write the hydrodynamic equation for the density

$$\partial_t\rho = -\partial_x j \tag{23}$$

$$j = \rho V\left(\frac{1}{\rho}\right) + \frac{1}{2}\partial_x V\left(\frac{1}{\rho}\right). \tag{24}$$

7. * We expand the current with respect to the small parameter η. We use, up to order η^2

$$\frac{1}{\rho} = \frac{1}{\bar{\rho}+\phi} = \frac{1}{\bar{\rho}} - \frac{\phi}{\bar{\rho}^2} + \frac{\phi^2}{\bar{\rho}^3}, \tag{25}$$

$$V\left(\frac{1}{\rho}\right) = \bar{V} - \frac{\phi}{\bar{\rho}^2}\bar{V}' + \frac{\phi^2}{\bar{\rho}^3}\left(\bar{V}' + \frac{\bar{V}''}{2\bar{\rho}}\right), \tag{26}$$

where $\bar{V}^{(n)} = V^{(n)}(1/\bar{\rho})$. For the second term of the current, we need only these expressions to order ϕ since $\partial_x \sim \eta$. We end up with the following expression for the current:

$$j = \bar{\rho}\bar{V} + \left(\bar{V} - \frac{\bar{V}'}{\bar{\rho}}\right)\phi + \frac{\bar{V}''}{2\bar{\rho}^3}\phi^2 - \frac{\bar{V}'}{2\bar{\rho}}\partial_x\phi. \tag{27}$$

Inserting in the conservation equation (23), we find

$$\partial_t\phi = -\partial_x\left[\left(\bar{V} - \frac{\bar{V}'}{\bar{\rho}}\right)\phi + \frac{\bar{V}''}{2\bar{\rho}^3}\phi^2 - \frac{\bar{V}'}{2\bar{\rho}}\partial_x\phi\right]. \tag{28}$$

8. The first term in brackets can be eliminated by a Galilean tranform, introducing $\tilde{\phi}(x,t) = \phi(x+vt,t)$, such that

$$\partial_t\tilde{\phi}(x,t) = v\partial_x\phi(x-vt,t) + \partial_t\phi(x-vt,t); \tag{29}$$

taking $v = \bar{V} - \frac{\bar{V}'}{\bar{\rho}}$, the equation in the moving frame is, discarding the tilde,

$$\partial_t\phi = \frac{\bar{V}'}{2\bar{\rho}}\partial_x^2\phi - \frac{\bar{V}''}{\bar{\rho}^3}\phi\partial_x\phi, \tag{30}$$

which is the Burgers equation.

9. Discarding the quadratic term in the equation for $\phi(x,t)$, we obtain

$$\partial_t\phi = \frac{\bar{V}'}{2\bar{\rho}}\partial_x^2\phi. \tag{31}$$

Given that the Laplacian has negative eigenvalues, the homogeneous state is stable for $\bar{V}' > 0$, which corresponds to the result of the first question.

10. We write the Burgers equation as

$$\partial_t\phi = D\partial_x^2\phi - a\partial_x\phi^2, \tag{32}$$

Inserting a travelling solution in the Burgers equation leads to

$$-vf' = Df'' - a(f^2)'. \tag{33}$$

Integrating, we get

$$Df'(x) = J - vf(x) + af(x)^2 = -\partial_f U(f(x)), \tag{34}$$

where we have introduced the potential

$$U(f) = -Jf + \frac{v}{2}f^2 - \frac{a}{3}f^3. \tag{35}$$

We could solve directly the ODE for $f(x)$, noting that the solution is of the form $f(x) = \alpha + \beta\tanh(\gamma x)$. Another route, which does not require to solve the ODE, is to use the fact that the "trajectory" $f(x)$ should connect two stationnary points of the potential $U(f)$, hence $U'(\phi_\pm) = 0$, setting

$$\phi^2 - \frac{v}{a}\phi + \frac{J}{a} = 0. \tag{36}$$

The second and third coefficients are the sum and product of the roots, meaning that

$$\phi_+ + \phi_- = \frac{v}{a}, \tag{37}$$

$$\phi_+\phi_- = \frac{J}{a}, \tag{38}$$

which gives the velocity of the "domain wall" as a function of the densities on both sides. It can be related to the result of the class, introducing the current $j = a\phi^2$:

$$\frac{v}{a} = \phi_+ + \phi_- = \frac{\phi_+^2 - \phi_-^2}{\phi_+ - \phi_-} = \frac{j_+ - j_-}{a(\phi_+ - \phi_-)}. \tag{39}$$

2 Hydraulic jump

11. Given that the jump is stationary, mass conservation imposes $h(x)U(x) = Q$, or $hU = h'U' = Q$.

12. * Momentum conservation relates the flux of momentum going out of a domain and the forces exerted on this domain. A unit volume of fluid at position x carries a momentum $\rho U(x)$ and the volume of fluid crossing a vertical section at x is $h(x)U(x)$ per unit width. The flux of momentum accross a section at x is thus $\rho h(x)U(x)^2$. The flux of momentum flowing out of a rectangular domain containing the jump is thus $\rho(h'U'^2 - hU^2)$.

On the other hand, the pressure is given by $p(x, z) = \rho g[h(x) - z]$; the effect of the atmospheric pressure p_0 on the domain that we consider vanishes, so that we do not have to consider it here. Integrating the pressure on both sides, we find that the total force exerted accross a section per unit width is $\rho g h(x)^2/2$, so that finally momentum conservation reads

$$h'U'^2 - hU^2 = \frac{g}{2}(h^2 - h'^2). \tag{40}$$

Using the previous relation to eliminate U, we find

$$\frac{2Q^2}{g}\left(\frac{1}{h'} - \frac{1}{h}\right) = h^2 - h'^2. \tag{41}$$

Simplifying, we arrive at

$$hh'(h + h') = \frac{2Q^2}{g}. \tag{42}$$

Solving this equation, we find that

$$\frac{h'}{h} = \frac{\sqrt{1 + 8\mathrm{Fr}^2} - 1}{2}, \tag{43}$$

where the Froude number is defined by

$$\mathrm{Fr} = \frac{U}{gh}. \tag{44}$$