Ecole Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Blood pressure waves (solution)

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1 Blood pressure waves

1. The fluid current is given by vA, so that the conservation of the fluid reads

$$\partial_t A + \partial_x(vA) = 0. \tag{1}$$

2. Neglecting the viscosity of the fluid, we obtain that the flow is a plug flow (the velocity of the fluid does not depend on the position in the cross-section) and that the velocity follows the Euler equation:

$$\partial_t v + v\partial_x v = -\rho^{-1}\partial_x p. \tag{2}$$

3. We consider an element of the artery of extent δx and $\delta\theta$. Its mass is $\rho_0 h r_0 \delta x \delta\theta$, so that its radial acceleration is $\rho_0 h r_0 \delta x \delta\theta \partial_{tt} r(x,t)$. We now evaluate the forces on this element. The radial pressure force is $r(x,t)\delta x\delta\theta p(x,t)$. A radius change from r_0 to r generates a strain $(r-r_0)/r$ and a stress $E(r-r_0)/r$. Projecting the stress on the radial direction, the elastic force is $-Eh(r-r_0)\delta x\delta\theta/r_0$. The Newton's second law thus reads

$$\rho_0 h r_0 \partial_{tt} r(x,t) = p(x,t)r(x,t) - Eh\frac{r(x,t)-r_0}{r_0}. \tag{3}$$

Assuming that $pr_0 \ll Eh$ means that the deformations $(r-r_0)/r$ remain small, which is the assumption of linear elasticity. In this case, we can assume that $p(x,t)r(x,t) \simeq r_0 p(x,t)$.

Last, we can translate the equation for the radius in an equation for the section $A(x,t) = \pi r(x,t)^2$. Due to small variations, $A(x,t) - A_0 \simeq 2\pi r_0[r(x,t)-r_0]$. We arrive at

$$\partial_{tt} A = \frac{2\pi r_0}{\rho_0 h} p - \frac{\pi E}{\rho_0 A_0}(A - A_0). \tag{4}$$

4. We start with the second law of motion. From the term linear in A on the r.h.s, we should have $\partial_{tt} = (\bar{t}/t)^2\partial_{\bar{t}\bar{t}} = [\pi E/(\rho_0 A_0)]\partial_{\bar{t}\bar{t}}$, hence $\bar{t}/t = \sqrt{\pi E/(\rho_0 A_0)}$. Then, the constant term on the r.h.s. imposes $\bar{A} = A/A_0$. With these two rescalings, the equation reads

$$\partial_{\bar{t}\bar{t}}\bar{A} = \frac{2r_0}{Eh}p + 1 - \bar{A}, \tag{5}$$

so that we have to define $\bar{p} = 2r_0 p/(Eh)$.

Second, we use the Euler equation (Eq. (2)). Without knowing the rescaling of the variable x, we can compare the second term on the l.h.s. and the r.h.s. to obtain $v_0 = \bar{v}(x,t)/v(x,t)$: $v_0^2 = p(x,t)/[\rho\bar{p}(x,t)]$, hence $v_0 = \sqrt{Eh/(2\rho r_0)}$. Last, using $x = \ell\bar{x}$, so that $\partial_x = \ell^{-1}\partial_{\bar{x}}$, the Euler equation becomes

$$\frac{\ell}{v_0}\sqrt{\frac{\pi E}{\rho_0 A_0}}\partial_{\bar{t}}\bar{v} + \bar{v}\partial_{\bar{x}}\bar{v} = -\partial_{\bar{x}}\bar{p}, \tag{6}$$

so that

$$\ell = \sqrt{\frac{\rho_0 h r_0}{2\rho}}. \tag{7}$$

This length scales as $\ell \sim \sqrt{hr_0}$, which is the geometric mean of the two lengths that define the artery. Injecting this variable change in the conservation equation (Eq. (1)) gives the dimensionless equation

$$\partial_{\bar{t}}\bar{A} + \partial_{\bar{x}}(\bar{v}\bar{A}) = 0. \tag{8}$$

5. The rest state of the system is given by $A_0 = 1$, $p_0 = 0$, $v_0 = 0$ in dimensionless variables. We write the quantities as $a = \sum_{n\geq 0} \epsilon^n a_n$. To the order ϵ, the equations become

$$\partial_{tt}A_1 + A_1 = p_1, \tag{9}$$
$$\partial_t v_1 = -\partial_x p_1, \tag{10}$$
$$\partial_t A_1 = -\partial_x v_1. \tag{11}$$

We look for solutions under the form $a_1(x,t) = \mathrm{e}^{\mathrm{i}(qx-\omega t)}\tilde{a}_1$. The equations become

$$(1-\omega^2)\tilde{A}_1 = \tilde{p}_1, \tag{12}$$
$$\omega\tilde{v}_1 = q\tilde{p}_1, \tag{13}$$
$$\omega\tilde{A}_1 = q\tilde{v}_1. \tag{14}$$

We end up with $1-\omega^2 = \omega^2/q^2$, so that the dispersion relation is

$$\omega = \frac{q}{1+q^2}. \tag{15}$$

In the long wavelength limit, $q \to 0$, the dispersion relation becomes $\omega = q$, so that the phase speed is $v_\phi = \omega/q = 1$. In this case, we also have $A_1 = p_1 = v_1$.

6. First, we note that $\partial_x = (\partial_x y)\partial_y = \epsilon^\chi \partial_y$ and $\partial_t = (\partial_t y)\partial_y + (\partial_t s)\partial_s = -\epsilon^\chi\partial_y + \epsilon^\tau\partial_s$. Inserting these relations in the general equations leads to

$$(\epsilon^\tau\partial_s - \epsilon^\chi\partial_y)^2 A' + A' = p, \tag{16}$$
$$(\epsilon^{\tau-\chi}\partial_s - \partial_y)v + v\partial_y v = -\partial_y p, \tag{17}$$
$$(\epsilon^{\tau-\chi}\partial_s - \partial_y)A' + \partial_y[(1+A')v] = 0, \tag{18}$$

where $A' = A - 1 = \sum_{n\geq 1}\epsilon^n A_n$. At order ϵ, we find that $A_1 = p_1 = v_1$ is a solution provided that $\tau > \chi$. Going to the order ϵ^ζ with $\zeta > 1$, these equations become

$$(\epsilon^\tau\partial_s - \epsilon^\chi\partial_y)^2 A_1 + A' = -p, \tag{19}$$
$$(\epsilon^{\tau-\chi}\partial_s - \partial_y)v + v\partial_y v = -\partial_y p, \tag{20}$$
$$(\epsilon^{\tau-\chi}\partial_s - \partial_y)A' + \partial_y[(1+A')v] = 0, \tag{21}$$

7. Going to the order ϵ^ζ with $\zeta > 1$, these equations become

$$\epsilon^{2\chi-1}\partial_{yy}A_1 = p_2 - A_2, \tag{22}$$
$$\epsilon^{\tau-\chi-1}\partial_s v_1 + v_1\partial_y v_1 = \partial_y v_2 - \partial_y p_2, \tag{23}$$
$$\epsilon^{\tau-\chi-1}\partial_s A_1 + \partial_y(A_1 v_1) = \partial_y A_2 - \partial_y v_2, \tag{24}$$

The ϵ factors cancel if $\chi = 1/2$ and $\tau = 3/2$. Deriving the first equation with respect to y and summing the two last equations, we arrive at

$$\partial_{yyy}A_1 = -\partial_s v_1 - v_1\partial_y v_1 - \partial_s A_1 - \partial_y(A_1 v_1). \tag{25}$$

Using that $A_1 = v_1$, we finally get

$$\partial_s A_1 + \frac{1}{2}\partial_{yyy}A_1 + \frac{3}{2}A_1\partial_s A_1 = 0, \tag{26}$$

which is the Korteweg-de Vries equation.

2 Dispersion in optical fibers

8. The nonlinear term comes from the nonlinearity of the medium, which is responsible, for instance, for the Kerr effect. For the linear terms, we consider the propagation of a wave with a dispersion relation $q(\omega)$; the amplitude of the scalar field $\psi(x,t)$ composed of pulsations $\omega = \omega_0 + \Omega$ is thus given by

$$\psi(x,t) = \mathrm{e}^{-\mathrm{i}\omega_0 t} \int \tilde{\psi}(\Omega) \mathrm{e}^{\mathrm{i}[q(\omega+\Omega)x - \Omega t]} \mathrm{d}\Omega. \tag{27}$$

Expanding

$$q(\omega_0 + \Omega) \simeq q_0 + q'\Omega + \frac{q''}{2}\Omega^2, \tag{28}$$

we get

$$\psi(x,t) = \mathrm{e}^{\mathrm{i}(q_0 x - \omega_0 t)} \int \tilde{\psi}(\Omega) \mathrm{e}^{\mathrm{i}\left[\left(q'\Omega + \frac{q''}{2}\Omega^2\right)x - \Omega t\right]} \mathrm{d}\Omega. \tag{29}$$

We absorb the prefactor in the "amplitude" $A(x,t) = \mathrm{e}^{-\mathrm{i}(q_0 x - \omega_0 t)}\psi(x,t)$. The derivatives of the amplitude read

$$\partial_x A(x,t) = \int \mathrm{i}\left(q'\Omega + \frac{q''}{2}\Omega^2\right) \tilde{\psi}(\Omega) \mathrm{e}^{\mathrm{i}\left[\left(q'\Omega + \frac{q''}{2}\Omega^2\right)x - \Omega t\right]} \mathrm{d}\Omega, \tag{30}$$

$$\partial_t A(x,t) = -\int \mathrm{i}\Omega\tilde{\psi}(\Omega) \mathrm{e}^{\mathrm{i}\left[\left(q'\Omega + \frac{q''}{2}\Omega^2\right)x - \Omega t\right]} \mathrm{d}\Omega, \tag{31}$$

$$\partial_{tt} A(x,t) = -\int \Omega^2\tilde{\psi}(\Omega) \mathrm{e}^{\mathrm{i}\left[\left(q'\Omega + \frac{q''}{2}\Omega^2\right)x - \Omega t\right]} \mathrm{d}\Omega. \tag{32}$$

Combining these equations, we find that

$$\partial_x A(x,t) + q'\partial_t A(x,t) + \frac{\mathrm{i}q''}{2}\partial_{tt} A(x,t) = 0. \tag{33}$$

$v_g = 1/q'$ is the group velocity.

9. Defining $t' = t - x/v_g$ and $x' = x$, we find that $\partial_x = \partial_{x'} - v_g^{-1}\partial_{t'}$ and $\partial_t = \partial_{t'}$. With these variables,

$$\partial_{x'} A(x',t') + \frac{\mathrm{i}q''}{2}\partial_{t't'} A(x',t') = 0. \tag{34}$$

This is a diffusion equation, with the variables exchanged with respect to the usual ones, and with an imaginary diffusion coefficient. We drop the primes in the following.

10. As for the diffusion equation, we use the Fourier transform, which we define as

$$A(x,t) = \int \tilde{A}(x,\omega) \mathrm{e}^{\mathrm{i}\omega t} \frac{\mathrm{d}\omega}{2\pi}, \tag{35}$$

$$\tilde{A}(x,\omega) = \int A(x,t) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t. \tag{36}$$

The Fourier transform of the wave packet at $x = 0$ is

$$\tilde{A}(0,\omega) = \sqrt{2\pi} t_0 \mathrm{e}^{-t_0^2\omega^2/2}. \tag{37}$$

Fourier transforming the propagation equation (34), we get

$$\partial_x \tilde{A}(x,\omega)) = \frac{\mathrm{i}q''\omega^2}{2}\tilde{A}(x,\omega)). \tag{38}$$

Integrating this ODE, we get

$$\tilde{A}(x,\omega) = \mathrm{e}^{\mathrm{i}q''\omega^2 x/2}\tilde{A}(0,\omega) = \sqrt{2\pi} t_0 \exp\left(-\frac{1}{2}\left[t_0^2 - \mathrm{i}q''x\right]\omega^2\right) \tag{39}$$

Inverting the Fourier transform and using the given formula, we get

$$A(x,t) = \frac{1}{\sqrt{1 - \frac{\mathrm{i}q''x}{t_0^2}}} \exp\left(-\frac{t^2}{2[t_0^2 - \mathrm{i}q''x]}\right). \tag{40}$$

11. The amplitude of the peak is given by the norm of the prefactor, which is now complex:

$$|A(x,0)| = \left(1 + \frac{q''^2 x^2}{t_0^4}\right)^{-1/4} \underset{x\to\infty}{\sim} x^{-1/2}. \tag{41}$$

The width $w(x)$ is given by the norm of the denominator in the exponential:

$$w(x) = \sqrt{t_0^4 + q''^2 x^2}. \tag{42}$$

Last, the imaginary term in the denominator in the exponential shows that the packet oscillates.