Ecole Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Modulational instability with the Nonlinear Schrödinger Equation

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In this tutorial, we study the stability of the plane wave solutions of the Nonlinear Schrödinger Equation (NLS), which we write as

$$\mathrm{i}\partial_t A = P\partial_{xx}A + Q|A|^2 A, \tag{1}$$

for the complex valued function $A(x,t)$.

1. When can bound states exist in the Schrödinger equation in a localized potential $V(x)$? Explain why the case $PQ > 0$ is called "focusing" and the case $PQ < 0$ is called "defocusing".

We now write $A(x,t) = \rho(x,t)\mathrm{e}^{\mathrm{i}\theta(x,t)}$, where $\rho(x,t)$ and $\theta(x,t)$ are the real amplitude and phase of $A(x,t)$.

2. * Write the coupled equations for $\partial_t\rho$ and $\partial_t\theta$.

3. Show the existence of a two-parameter family of solutions with constant amplitude, $\rho(x,t) = \rho_0$ and describe these solutions.

4. * Determine the stability of the plane wave solutions by introducing a small perturbation $\rho_1(x,t)$ and $\theta_1(x,t)$. You can look for plane wave solutions: $a(x,t) = \bar{a}\mathrm{e}^{\sigma t + \mathrm{i}px}$, where $a \in \{\rho_1, \theta_1\}$.