Ecole Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Modulational instability with the Nonlinear Schrödinger Equation (solution)

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1. The Schrödinger equation in a potential $V(x)$ takes the form

$$\mathrm{i}\partial_t\psi(x,t) = -\partial_{xx}\psi(x,t) + V(x)\psi(x,t). \tag{1}$$

Bound states can exist only if $V(x) < 0$; in this case the prefactors of the two terms in the r.h.s. have the same sign, corresponding to the case $PQ > 0$ for the NLS, which is called "focusing". In the NLS, it is the modulus of the wavefunction that plays the role of the potential, $V(x) \approx Q|\psi(x,t)|^2$.

2. Using $A(x,t) = \rho(x,t)\mathrm{e}^{\mathrm{i}\theta(x,t)}$, the derivatives read

$$\partial_t A = (\partial_t\rho + \mathrm{i}\rho\partial_t\theta)\,\mathrm{e}^{\mathrm{i}\theta}, \tag{2}$$

$$\partial_{xx}A = \left[\partial_{xx}\rho + 2\mathrm{i}\partial_x\rho\partial_x\theta + \mathrm{i}\rho\partial_{xx}\theta - \rho(\partial_x\theta)^2\right]\mathrm{e}^{\mathrm{i}\theta}. \tag{3}$$

The nonlinear term is simply $|A|^2A = \rho^3\mathrm{e}^{\mathrm{i}\theta}$.

Plugging these expressions in the NLS and identifying the real and imaginary parts, we find

$$\partial_t\rho = P(2\partial_x\rho\partial_x\theta + \rho\partial_{xx}\theta), \tag{4}$$

$$\partial_t\theta = -P\frac{\partial_{xx}\rho}{\rho} - Q\rho^2 + P(\partial_x\theta)^2. \tag{5}$$

3. With a constant amplitude $\rho_0 > 0$, the equations for ρ and θ reduce to

$$\partial_{xx}\theta = 0, \tag{6}$$

$$\partial_t\theta = -Q\rho_0^2 + P(\partial_x\theta)^2. \tag{7}$$

From the first equation we deduce that $\theta(x,t) = q(t)x + \theta_0(t)$. The second equation then gives $\dot{q}x + \dot{\theta}_0 = Pq^2 - Q\rho_0^2$, leading to $\dot{q} = 0$ and $\omega = \dot{\theta} = Pq^2 - Q\rho_0^2$. These are plane wave solutions. The dispersion relation gives the temporal dependence for a wavevector q and an amplitude ρ_0.

4. We now write $\rho = \rho_0 + \epsilon\rho_1$ and $\theta = \theta_0 + \epsilon\theta_1$. Writing the evolution equations of ρ and θ at the order ϵ, we find

$$\partial_t\rho_1 = P(2\partial_x\theta_0\partial_x\rho_1 + \rho_0\partial_{xx}\theta_1) = P(2q\partial_x\rho_1 + \rho_0\partial_{xx}\theta_1), \tag{8}$$

$$\partial_t\theta_1 = -\frac{P}{\rho_0}\partial_{xx}\rho_1 - 2Q\rho_0\rho_1 + 2Pq\partial_x\theta_1. \tag{9}$$

Looking for plane wave solutions of the form $a(x,t) = \bar{a}\mathrm{e}^{\sigma t + \mathrm{i}px}$, these equation become

$$\sigma\begin{pmatrix}\bar{\rho}_1\\ \bar{\theta}_1\end{pmatrix} = \begin{pmatrix}2\mathrm{i}Ppq & -P\rho_0p^2\\ \frac{P}{\rho_0}p^2 - 2Q\rho_0 & 2\mathrm{i}Ppq\end{pmatrix}\begin{pmatrix}\bar{\rho}_1\\ \bar{\theta}_1\end{pmatrix}. \tag{10}$$

This is an eigenvalue equation for σ, which should satisfy:

$$(\sigma - 2\mathrm{i}Ppq)^2 = p^2\left(2PQ\rho_0^2 - P^2p^2\right). \tag{11}$$

The growth rate acquires a real part if the r.h.s. is positive. In this case, one of the two solutions for σ has a positive real part and the plane wave is unstable. This is the case as soon as $PQ > 0$ because the wavevector of the instability p can be as small as we want.