École Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Rayleigh-Bénard instability

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1 Rayleigh-Bénard instability

In this tutorial, we study the onset of the Rayleigh-Bénard instability in a fluid under a temperature gradient. We consider a fluid between two flat and horizontal plates at a distance d and maintained at temperatures $\Theta_\downarrow$ and $\Theta_\uparrow = \Theta_\downarrow - d\vartheta$. When the fluid is at rest, the thermal transfer between the plates occurs only via diffusion, with a heat diffusion coefficient κ. When the bottom plate is hotter than the top plate, the fluid close to the bottom plate is lighter than the fluid above it due to its thermal expansion, which may be enough to induce convection rolls in the fluid: this is the Rayleigh-Bénard instability.

The fluid has a kinematic viscosity ν, an average density $\bar{\rho}$ and a thermal expansion coefficient α, so that the density at temperature θ is $\rho = \bar{\rho}[1-\alpha(\theta-\bar{\theta})]$. We use the Boussinesq approximation where the density of the fluid is assumed to be constant ($\rho = \bar{\rho}$), so that the flow is incompressible, except when it is multiplied by the gravity acceleration along z.

The equations describing the velocity of the fluid $\boldsymbol{v}(\boldsymbol{r},t)$, its temperature $\theta(\boldsymbol{r},t)$, its pressure $p(\boldsymbol{r},t)$, are

$$\nabla \cdot \boldsymbol{v} = 0, \tag{1}$$

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = \nu\nabla^2\boldsymbol{v} - \bar{\rho}^{-1}\nabla p - \left[1-\alpha\left(\theta-\bar{\theta}\right)\right] g\hat{\boldsymbol{e}}_z, \tag{2}$$

$$\partial_t\theta + \boldsymbol{v}\cdot\nabla\theta = \kappa\nabla^2\theta. \tag{3}$$

1. Describe the rest state of the system ($\boldsymbol{v} = 0$).

2. Discuss the terms that favor or penalize the instability.

3. Linearize the equations around the rest state to obtain the equations for the perturbation, $(\boldsymbol{v}(\boldsymbol{r},t), \theta_1(\boldsymbol{r},t), p_1(\boldsymbol{r},t))$. What are the boundary conditions?

We now assume that the component of the velocity in the y direction, v_y, is zero, and that the fields do not depend on y. We also follow Rayleigh and use a free slip boundary condition along the walls.

4. Fourier transform the linearized equations with respect to the space variable x, with a wavevector q, and assume an exponential growth with rate σ: write the new equations and give the boundary conditions. You can introduce the differential operator $\hat{q}^2 = q^2 - \partial_{zz}$.

5. * Determine a closed equation for θ_1.

6. Show that a z-dependence as $\sin(\pi z/d)$ for v_z and θ_1 satisfies all the boundary conditions. Justify the use of this ansatz; we will use it in the following.

7. Determine the dispersion relation, the instability threshold and the critical wavenumber. Write them in dimensionless form with the Rayleigh number $\mathrm{Ra} = \alpha g\vartheta d^4/(\nu\kappa)$.

8. Sketch the flow $\boldsymbol{v}$ above the instability threshold.

9. Above what temperature difference $\Delta\Theta$ are convection rolls observed in a hot water pot? We give $\alpha \simeq 5 \times 10^{-4}\,\mathrm{K}^{-1}$, $\kappa \simeq 0.6\,\mathrm{W\,m^{-1}\,K}$, $\nu \simeq 9\times10^{-7}\,\mathrm{m^2/s}$.

2 Stability analysis of the Schnackenberg model

The Schnackenberg model consider the following system of reactions for the species A, B, C, X, Y:

$$A \xrightarrow{k_1} X, \tag{4}$$

$$X \xrightarrow{k_2} C, \tag{5}$$

$$2X + Y \xrightarrow{k_3} 3X, \tag{6}$$

$$B \xrightarrow{k_4} Y. \tag{7}$$

The constants k_i are the prefactors of the reaction terms. We assume that the concentrations of A and B are fixed by some external process.

1. Write the reaction diffusion equations for the species X and Y.

2. Normalize the equations to obtain

$$\partial_t u = a - u + u^2 v + \partial_{xx} u, \tag{8}$$

$$\partial_t v = b - u^2 v + d\partial_{xx} v, \tag{9}$$

where $u(x,t)$ and $v(x,t)$ are the normalized concentrations of the species X and Y and a, b, d are coefficients.

3. What are the necessary conditions to observe a I-s Turing instability? Which species is the inhibitor?

4. Determine the instability threshold.