Ecole Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Rayleigh-Bénard instability (solution)

Vincent Démery, Olivier Pierre-Louis

1 Rayleigh-Bénard instability

1. When the fluid is at rest, the equations become

$$\partial_z p = -\rho_0 g \left[1 - \alpha\left(\theta - \bar{\theta}\right)\right], \tag{1}$$

$$\nabla^2\theta = 0. \tag{2}$$

From the second equation we obtain the temperature profile

$$\theta = \Theta_\downarrow - \vartheta z. \tag{3}$$

Integrating the equation for the pressure, we get

$$p(z) = p_0 - \bar{\rho} g \int_0^z \left[1 - \alpha\left(\Theta_\downarrow - \vartheta z' - \bar{\theta}\right)\right] \mathrm{d}z' = p_0 - \bar{\rho} g \left(z\left[1 - \alpha\left(\Theta_\downarrow - \bar{\theta}\right)\right] + \frac{z^2}{2}\alpha\vartheta\right). \tag{4}$$

2. The buoyancy term should drive the instability; its intensity is given by αg and it is larger if the thermal gradient ϑ is larger. On the contrary, the "diffusive" terms ν and κ can penalize the instability.

3. Linearizing around the rest state, we find

$$\nabla \cdot \boldsymbol{v} = 0, \tag{5}$$

$$\partial_t \boldsymbol{v} - \nu\nabla^2\boldsymbol{v} = -\bar{\rho}^{-1}\nabla p_1 + \alpha g \theta_1 \hat{\boldsymbol{e}}_z, \tag{6}$$

$$\partial_t\theta_1 - \kappa\nabla^2\theta_1 = v_z\vartheta. \tag{7}$$

The boundary conditions are usually a no slip boundary condition at the walls. The temperature is imposed at the walls, hence the perturbation should vanish, hence

$$\boldsymbol{v}(z=0) = \boldsymbol{v}(z=d) = 0, \tag{8}$$

$$\theta_1(z=0) = \theta_1(z=d) = 0. \tag{9}$$

4. The equations now read

$$\mathrm{i}qv_x + \partial_z v_z = 0, \tag{10}$$

$$(\sigma + \nu\hat{q}^2)v_x = -\bar{\rho}^{-1}\mathrm{i}qp_1, \tag{11}$$

$$(\sigma + \nu\hat{q}^2)v_z = -\bar{\rho}^{-1}\partial_z p_1 + \alpha g\theta_1, \tag{12}$$

$$(\sigma + \kappa\hat{q}^2)\theta_1 = v_z\vartheta. \tag{13}$$

And the boundary conditions are

$$\partial_z v_x(0) = \partial_z v_x(d) = 0, \tag{14}$$

$$v_z(0) = v_z(d) = 0, \tag{15}$$

$$\theta_1(0) = \theta_1(d) = 0. \tag{16}$$

5. Combining Eqs. (10, 11) leads to

$$(\sigma + \nu\hat{q}^2)\partial_z v_z = -\bar{\rho}^{-1} q^2 p_1. \tag{17}$$

Using this relation to eliminate p_1 in Eq. (12) leads to

$$(\sigma + \nu\hat{q}^2)v_z = q^{-2}(\sigma + \nu\hat{q}^2)\partial_{zz} v_z + \alpha g \theta_1, \tag{18}$$

which, multiplying by q^2 and using $\hat{q}^2$, leads to

$$\hat{q}^2(\sigma + \nu\hat{q}^2)v_z = \alpha g q^2 \theta_1, \tag{19}$$

Finally, using this relation in Eq. (16), we find

$$\hat{q}^2(\sigma + \nu\hat{q}^2)(\sigma + \kappa\hat{q}^2)\theta_1 = \alpha g \vartheta q^2 \theta_1. \tag{20}$$

6. Taking $v_z, \theta_1 \propto \sin(\pi z/d)$ allows to satisfy the boundary conditions (15, 16). Incompressibility then gives $v_x \propto \cos(\pi z/d)$, which satisfies the boundary condition (14).

7. Now, the differential operator is $\hat{q}^2 = q^2 + (\pi/d)^2$, so that the dispersion relation can be identified in Eq. (20):

$$\hat{q}^2 \left(\sigma + \nu\hat{q}^2\right)\left(\sigma + \kappa\hat{q}^2\right) = \alpha g \vartheta q^2. \tag{21}$$

Expanding this relation to write it as a polynomial equation for σ gives

$$\sigma^2 + (\nu + \kappa)\hat{q}^2\sigma + \nu\kappa\hat{q}^4 - \frac{\alpha g \vartheta q^2}{\hat{q}^2} = 0. \tag{22}$$

It has a real positive solution for

$$\nu\kappa\hat{q}^4 - \frac{\alpha g \vartheta q^2}{\hat{q}^2} < 0, \tag{23}$$

hence for

$$\frac{\alpha g \vartheta}{\nu\kappa} > \frac{\hat{q}^6}{q^2} = \frac{\left(q^2 + \frac{\pi^2}{d^2}\right)^3}{q^2}. \tag{24}$$

We can introduce the Rayleigh number Ra and the normalized wavevector $\bar{q} = dq$,

$$\mathrm{Ra} = \frac{\alpha g \vartheta d^4}{\nu\kappa} > \frac{\hat{q}^6}{q^2} = \frac{\left(\bar{q}^2 + \pi^2\right)^3}{\bar{q}^2}. \tag{25}$$

The r.h.s. is minimal for $\bar{q} = \bar{q}^* = \pi/\sqrt{2}$, and its value is

$$\mathrm{Ra_c} = \frac{27\pi^4}{4} \simeq 658. \tag{26}$$

8. The flow field describes convection rolls.

9. For $d = 10\,\mathrm{cm}$, we find $\Delta\Theta_c \simeq 70\,\mathrm{K}$.

2 Stability analysis of the Schnackenberg model

1. The equation for the concentrations X and Y are

$$\partial_t X = k_1 A - k_2 X + k_3 X^2 Y + D_X \partial_{xx} X, \tag{27}$$
$$\partial_t Y = k_4 B - k_3 X^2 Y + D_Y \partial_{xx} Y. \tag{28}$$

2. Rescaling the variables as $t = \tau\bar{t}$, $x = \ell\bar{x}$, $X = c_X u$, $Y = c_Y v$, where τ, ℓ, c_X and c_Y are parameters, and choosing the parameters

$$\tau = k_2^{-1}, \tag{29}$$
$$\ell = \sqrt{D_X/k_2}, \tag{30}$$
$$c_X = c_Y = \sqrt{k_2/k_3}, \tag{31}$$

we find the proposed system of equations with

$$a = k_1 \sqrt{k_3} A / k_2^{3/2}, \tag{32}$$
$$b = k_4 \sqrt{k_3} B / k_2^{3/2}, \tag{33}$$
$$d = D_Y / D_X. \tag{34}$$

3. The homogeneous steady state is given by $a - u + u^2 v = 0$, $b - u^2 v = 0$, leading to $u = a + b$, $v = b/(a+b)^2$. Denoting this state as (u_0, v_0) and linearizing around it for the perturbation (u_1, v_1) leads to

$$\partial_t \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 2u_0 v_0 - 1 - q^2 & u_0^2 \\ -2u_0 v_0 & -u_0^2 - dq^2 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}. \tag{35}$$

We have written this equation in Fourier space with the wavevector q. The square matrix is of the form

$$\begin{pmatrix} g_{11} - D_1 q^2 & g_{12} \\ g_{21} & g_{22} - D_2 q^2 \end{pmatrix}. \tag{36}$$

The inhibitor is the element with a negative diagonal coefficient, which is B here.

The equation for the growth rate σ, which is an eigenvalue of this matrix, is

$$\begin{aligned} 0 &= \left(\sigma - g_{11} + D_1 q^2\right)\left(\sigma - g_{22} + D_2 q^2\right) - g_{12} g_{21} &(37)\\ &= \sigma^2 - \left[g_{11} + g_{22} - (D_1 + D_2) q^2\right] \sigma + \left(g_{11} - D_1 q^2\right)\left(g_{22} - D_2 q^2\right) - g_{12} g_{21} &(38)\\ &= \sigma^2 - S\sigma + P, &(39)\end{aligned}$$

where S and P are the sum and product of the eigenvalues.

For the system to develop a Turing I-s instability, the mode $q = 0$ should be stable, leading to $S_0 < 0$ and $P_0 > 0$:

$$0 > S_0 = g_{11} + g_{22} = \frac{b - a - (a+b)^3}{a+b}, \tag{40}$$
$$0 < P_0 = g_{11} g_{22} - g_{12} g_{21} = (a+b)^2. \tag{41}$$

The product is

$$P = P_0 - (g_{11} D_2 + g_{22} D_1) q^2 + D_1 D_2 q^4. \tag{42}$$

For this expression to turn negative for positive wavevectors, a necessary condition is $g_{11} D_2 + g_{22} D_1 > 0$, leading to $d > (a+b)^3/(b-a)$.

Finally, the three conditions reduce to

$$1 < \frac{(a+b)^3}{b-a} < d. \tag{43}$$

4. Considering the equation $P = 0$ as an equation for q, its discriminant is

$$\Delta = (g_{11} D_2 + g_{22} D_1)^2 - 4 P_0 D_1 D_2. \tag{44}$$

Considering the necessary condition $g_{11} D_2 + g_{22} D_1 > 0$, the condition $\Delta > 0$ amounts to

$$g_{11} D_2 + g_{22} D_1 > 2\sqrt{P_0 D_1 D_2}. \tag{45}$$

Replacing g_{ij} with its values, $D_1 = 1$ and $D_2 = d$, we obtain an equation for d:

$$d - \frac{2(a+b)^2}{b-a}\sqrt{d} - \frac{(a+b)^3}{b-a} > 0, \tag{46}$$

which is satisfied if

$$d > d_+ = \frac{(a+b)^4}{(b-a)^2}\left(1 + \sqrt{\frac{2b}{a+b}}\right)^2. \tag{47}$$