École Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Gradient dynamics, multi-scale expansion, blow-up (solution)

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1 Gradient dynamics for the anisotropic RGL equation

1. It can be shown easily that this equation derives from the functional

$$\mathcal{F}[A] = \int \left(\frac{1}{2} \left[|A(\boldsymbol{r},t)|^2 - 1 \right]^2 + \left| \left(\partial_x - \frac{\mathrm{i}}{2}\partial_{yy} \right) A(\boldsymbol{r},t) \right|^2 \right) \mathrm{d}\boldsymbol{r}. \tag{1}$$

2. First, we observe that $\mathcal{F}$ is bounded from below: $\mathcal{F}[A] \geq 0$. Second, we compute

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \int \left(\partial_t A \frac{\delta F}{\delta A} + \partial_t A^* \frac{\delta F}{\delta A^*} \right) \mathrm{d}\boldsymbol{r} = -2 \int |\partial_t A|^2 \mathrm{d}\boldsymbol{r} \leq 0; \tag{2}$$

hence, $\mathcal{F}$ is a Lyapunov functional.

2 Multi-scale expansion of model A

3. We first write explicitly the equation for u:

$$\partial_t u = \Gamma(u) \left[\beta \nabla^2 u - v'(u) \right]. \tag{3}$$

To the first order in u, $v'(u) = uv''(0)$ and $\Gamma(u) = \Gamma(0)$, so that

$$\partial_t u = \Gamma(0) \left[\beta \nabla^2 u - v''(0) u \right]. \tag{4}$$

In Fourier space, it reads

$$\partial_t u = -\Gamma(0) \left[\beta q^2 + v''(0) \right] u. \tag{5}$$

The state $u = 0$ is stable for all wavevectors q if $v''(0) > 0$.

4. All the parameters $\Gamma(0)$, β and a can be absorbed by a rescaling of the variables, so we are left with

$$\partial_t u = \nabla^2 u + \epsilon u. \tag{6}$$

Rescaling lengths and time with $t = \bar{t}/\epsilon$, $\boldsymbol{r} = \bar{\boldsymbol{r}}/\epsilon^{1/2}$, so that $\partial_t = \epsilon \partial_{\bar{t}}$ and $\nabla = \epsilon^{1/2}\bar{\nabla}$, we get

$$\partial_{\bar{t}} u = \bar{\nabla}^2 u + u. \tag{7}$$

5. Expanding $v(u)$ to the next order, the equation becomes

$$\partial_t u = \nabla^2 u + \epsilon u - g u^3. \tag{8}$$

Rescaling t and $\boldsymbol{r}$ as above and defining $u = \epsilon^\alpha \bar{u}$, we get

$$\epsilon^{1+\alpha} \partial_{\bar{t}} \bar{u} = \epsilon^{1+\alpha} \bar{\nabla}^2 \bar{u} + \epsilon^{1+\alpha} \bar{u} - \epsilon^{3\alpha} g \bar{u}^3. \tag{9}$$

The exponents should be equal, hence $\alpha = 1/2$. Note that a non-linear term of the form $u^2\nabla^2 u$ would be irrelevant in the limit $\epsilon \to 0$.

3 Blow-up in the Fisher-Kolmogorov equation

We consider the Fisher-Kolmogorov equation equation for a real field $u(\boldsymbol{r}, t)$ over a domain Ω with volume V, with a Neumann boundary condition $\hat{\boldsymbol{n}}(\boldsymbol{r}) \cdot \nabla u(\boldsymbol{r}, t) = 0$ for $\boldsymbol{r} \in \partial\Omega$, where $\hat{\boldsymbol{n}}(\boldsymbol{r})$ is a unit vector normal to the boundary:

$$\partial_t u = u + u^2 + \nabla^2 u. \tag{10}$$

We define the mass

$$m(t) = \frac{1}{V} \int_\Omega u(\boldsymbol{r}, t) \mathrm{d}\boldsymbol{r}. \tag{11}$$

6. Integrating the evolution equation for u, we get

$$\dot{m}(t) = m(t) + \frac{1}{V} \int u(\boldsymbol{r}, t)^2 \mathrm{d}\boldsymbol{r}. \tag{12}$$

The Laplacian integrates to zero due to the Neumann boundary condition. Introducing the scalar product

$$\langle f, g \rangle = \frac{1}{V} \int f(\boldsymbol{r}) g(\boldsymbol{r}) \mathrm{d}\boldsymbol{r}, \tag{13}$$

we see that the second term in the evolution of the mass can be writen as

$$\frac{1}{V} \int u(\boldsymbol{r}, t)^2 \mathrm{d}\boldsymbol{r} = \|u\|^2 = \|u\|^2 \|1\|^2 \geq \langle u, 1 \rangle^2 = m(t)^2. \tag{14}$$

7. If $m(0) > 0$, then $m(t) \geq n(t)$ where $n(0) = m(0)$ and

$$\dot{n}(t) = n(t) + n(t)^2. \tag{15}$$

We now solve this equation. First, we define $p(t) = \mathrm{e}^{-t} n(t)$, so that $\dot{p}(t) = \mathrm{e}^{-t}[-n(t) + \dot{n}(t)] = \mathrm{e}^{-t} n(t)^2 = \mathrm{e}^t p(t)^2$. Integrating $\dot{p}(t)/p(t)^2 = \mathrm{e}^t$ leads to $p(0)^{-1} - p(t)^{-1} = \mathrm{e}^t - 1$, so that $p(t) = \left[1 + p(0)^{-1} - \mathrm{e}^t\right]^{-1}$. Finally

$$n(t) = \frac{1}{\left[1 + p(0)^{-1}\right] \mathrm{e}^{-t} - 1}. \tag{16}$$

Defining $t_0 = \log\left(1 + p(0)^{-1}\right)$, it leads to

$$m(t) \geq n(t) = \frac{1}{\mathrm{e}^{t_0 - t} - 1}. \tag{17}$$

The lower bound diverges at $t = t_0$, hence the solution should diverge at last at t_0.

Note that the lower bound is reached if $u(\boldsymbol{r}, t)$ is uniform.