École Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Localized buckling of a floating sheet

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We study the buckling of a sheet placed at a liquid air interface and submitted to a uniaxial compression. It has been observed that uniform wrinkles form above the compression threshold, but that the buckling profile localizes as the compression is increased [1] (Fig. 1). Here, we analyse the buckling pattern beyond the linear instability using a multiple lengthscales expansion, following Ref. [2]. We note that an exact solution to the same problem has been found [3], which describes the buckling pattern as a soliton.

The shape of the sheet is described by the height function $h(s)$, where s is the curvilinear coordinate along the sheet. The dimensionless energy functional is

$$E[h(s)] = \int_{-L/2}^{L/2} e(h(s), h'(s), h''(s))\mathrm{d}s \tag{1}$$

where L is the length of the sheet, which is taken to infinity when convenient, and the energy density is

$$e(h, h', h'') = \frac{1}{2}\frac{h''^2}{1-h'^2} - P\left(1-\sqrt{1-h'^2}\right) + \frac{h^2}{2}\sqrt{1-h'^2}. \tag{2}$$

The first term is the bending energy, the second is the potential energy associated to the compression with "pressure" P, which is the control parameter, and the last is the gravitational energy of the deformed liquid interface.

1. * Perform the linear stability analysis of the flat interface.

Above the threshold, we define the small parameter $\epsilon = P_c - P$, where P_c is the critical pressure and we use the fact that the pressure decreases above threshold. We look for a solution of the form $h(s) = \epsilon^\alpha H(\epsilon^\beta s)\cos(k_c s)$, where α and β are to be determined, and $H(S)$ describes the enveloppe of the profile.

2. ** Expand the energy to the lowest order in ϵ and average it with respect to the "fast oscillations" ($\langle \cos(k_c s)^2\rangle = 1/2\dots$). Show that it can be written under the form

$$E[H(S)] = A\int\left[\frac{1}{2}H'(S)^2 - V(H(S))\right]\mathrm{d}S, \tag{3}$$

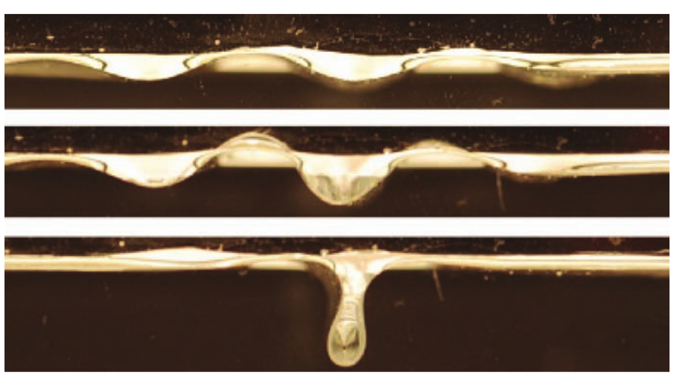

Figure 1: Side view of a compressed elastic sheet resting on a liquid [1]. The compression length increases from top to bottom.

where A is a constant and $V(H) = -\frac{1}{8}H^2 + \frac{1}{32}H^4$.

3. * Interpret this energy and the localized solutions that it can produce. Show that it has a solution of the form $H(S) = a/\cosh(bS)$. Write down the final form of the profile $h(s)$ and discuss it.

References

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