École Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Localized buckling of a floating sheet (solution)

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1. We first expand the energy at the lowest order in h ($\sim h^2$):

$$e(h,h',h'') = \frac{1}{2}\left(h''^2 - Ph'^2 + h^2\right). \tag{1}$$

In the infinite size limit, the energy can be written for the Fourier transform of the height, $\tilde{h}(k)$ (up to an unimportant factor):

$$E[h] = \int (k^4 - Pk^2 + 1)|\tilde{h}(k)|^2 \mathrm{d}k. \tag{2}$$

We see that the energy becomes negative for $P \geq P_c = 2$, for wavevectors around $k_c = 1$.

2. We start by expanding e to the fourth order in h, using $P = 2 - \epsilon$ and discarding the non-dominant terms such as ϵh^4:

$$e(h,h',h'') = \frac{1}{2}\left(h''^2 - 2h'^2 + h^2\right) + \frac{\epsilon}{2}h'^2 + \frac{1}{2}h'^2h''^2 - \frac{1}{4}h'^4 - \frac{1}{4}h^2h'^2. \tag{3}$$

With the proposed form, $h(s) = \epsilon^\alpha H(\epsilon^\beta s)\cos(s)$, the derivatives are

$$h'(s) = \epsilon^\alpha\left[-H(\epsilon^\beta s)\sin(s) + \epsilon^\beta H'(\epsilon^\beta s)\cos(s)\right], \tag{4}$$

$$h''(s) = \epsilon^\alpha\left[-H(\epsilon^\beta s)\cos(s) - 2\epsilon^\beta H'(\epsilon^\beta s)\sin(s) + \epsilon^{2\beta}H''(\epsilon^\beta s)\cos(s)\right]. \tag{5}$$

To simplify, we average over the fast oscillations:

$$\langle h^2\rangle = \frac{\epsilon^{2\alpha}}{2}H(S)^2, \tag{6}$$

$$\langle h'^2\rangle = \frac{\epsilon^{2\alpha}}{2}\left[H(S)^2 + \epsilon^{2\beta}H'(S)^2\right], \tag{7}$$

$$\langle h''^2\rangle = \frac{\epsilon^{2\alpha}}{2}\left[H(S)^2 + \epsilon^{2\beta}\left[4H'(S)^2 - 2H(S)H''(S)\right] + \epsilon^{4\beta}H''(S)^2\right]. \tag{8}$$

For the terms of order h^4, keeping only the dominant terms, we have

$$\langle h^2h'^2\rangle = \frac{1}{8}\epsilon^{4\alpha}H^4, \tag{9}$$

$$\langle h'^4\rangle = \frac{3}{8}\epsilon^{4\alpha}H^4, \tag{10}$$

$$\langle h'^2h''^2\rangle = \frac{1}{8}\epsilon^{4\alpha}H^4. \tag{11}$$

In Eq. (3), we find that the dominant terms, of order $\epsilon^{2\alpha}$, cancel, and that the next order terms are

$$\langle e\rangle = \frac{\epsilon^{2\alpha+2\beta}}{2}(H'^2 - HH'') + \frac{\epsilon^{1+2\alpha}}{4}H^2 - \frac{1}{16}\epsilon^{4\alpha}H^4. \tag{12}$$

As shown by the following analysis, all the terms are needed, which imposes

$$\alpha = \beta = \frac{1}{2}. \tag{13}$$

At order ϵ^2, the normalized energy is

$$\bar{e}(H, H', H'') = \epsilon^{-2}\langle e\rangle = \frac{1}{4}H^2 - \frac{1}{16}H^4 + \frac{1}{2}(H'^2 - HH''). \tag{14}$$

It can be written as

$$\bar{e} = 2\left[\frac{1}{2}H'^2 - V(H)\right] - \frac{1}{2}(HH')', \tag{15}$$

with $V(H) = -\frac{1}{8}H^2 + \frac{1}{32}H^4$. The total derivative just contributes a boundary term and we discard it; this can be justified later. We end up with the energy

$$E[H(S)] = A\int L(H(S), H'(S))\mathrm{d}S. \tag{16}$$

where we have defined the Lagrangian

$$L(H, H') = \frac{1}{2}H'^2 - V(H). \tag{17}$$

3. The Lagrangian (17) is the Lagrangian of classical mechanics for a potential $V(H)$. The potential as a local maximum at $H = 0$, with $V(0) = 0$, minima at $H = \pm\sqrt{2}$, and we note that $V(\pm 2) = 0$.

Looking for localized buckling, the trajectory should start at $H = 0$ for $S \to -\infty$, travel to $H = \pm 2$ and come back to $H = 0$ for $S \to \infty$. This corresponds to an energy $\mathcal{E} = \frac{1}{2}H'^2 + V(H) = 0$, and thus to the equation

$$H' = \pm\sqrt{-2V(H)} = \pm\sqrt{\frac{H^2}{4} - \frac{H^4}{16}}. \tag{18}$$

Looking for a solution of the form $H(S) = a/\cosh(bS)$, we find that we should have $a = 2$ and $b = 1/2$. As a consequence, the form of the deformation is

$$h(s) = \frac{2\epsilon^{1/2}}{\cosh(\epsilon^{1/2}s/2)}\cos(s). \tag{19}$$

We see that as the buckling pattern grows, it becomes more and more localized.