École Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Collapse of domains in one and two dimensions

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1 Collapse of a 1D domain in the Cahn-Allen equation

We consider a field $u(x,t)$ obeying the Cahn-Allen equation

$$\partial_t u = u - u^3 + \partial_{xx} u. \tag{1}$$

1. Check that the profile

$$u_k(x) = \tanh\left(\frac{x}{\sqrt{2}}\right) \tag{2}$$

is stationnary under the Cahn-Allen equation.

We wish to analyze the evolution of the size $2x_0(t)$ of an isolated domain. We describe this domain with an approximate profile of the form, for $x_0(t) \gg 1$,

$$u(x,t) = u_k(x + x_0(t)) - u_k(x - x_0(t)) - 1. \tag{3}$$

2. Insert the ansatz (3) in the evolution equation and show that

$$\dot{x}_0 \left(2 - u_+^2 - u_-^2\right) = 3\sqrt{2}(u_+ - u_-)(u_+ - 1)(u_- + 1), \tag{4}$$

where $u_\nu = u_k(x + \nu x_0(t))$ where $\nu \in \{-1, 1\}$.

3. Evaluate Eq. (4) at $x = x_0(t)$ and obtain the dynamics of $x_0(t)$.

4. Solve the dynamics of $x_0(t)$ and find the collapse time t_c when $x_0 = 0$.

2 Growth or collapse of a 2D circular domain

We consider a domain where the front moves with a velocity $c = c_* - D\kappa$ along its normal, where κ is its local curvature; this is the Eikonal equation.

5. Give the evolution of the radius $r(t)$ of a circular domain. Introduce a critical radius r_* and describe qualitatively the dynamics as a function of $r(0)$.

6. Find an implicit expression for the radius $r(t)$ for $r(0) < r_*$. We give

$$\int_{r_1}^{r_2} \frac{r\mathrm{d}r}{r_* - r} = r_* \log\left(\frac{r_* - r_1}{r_* - r_2}\right) - r_2 + r_1. \tag{5}$$

7. Determine the collapse time. What do you obtain in the limit $c_* \to 0$? Discuss this result and compare it to the one-dimensional case.

8. Determine the dynamics of a small deviation $h(x,t)$ to a flat front corresponding to the Eikonal equation, up to the order h^2.