École Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Collapse of domains in one and two dimensions (solution)

Vincent Démery, Olivier Pierre-Louis

1 Collapse of a 1D domain in the Cahn-Allen equation

1. We just need $\tanh' = 1 - \tanh^2$ and $\tanh'' = -2\tanh(1-\tanh^2)$.

2. Inserting the ansatz in the equation, we find

$$\partial_t u = \frac{\dot{x}_0}{\sqrt{2}}\left(2 - u_+^2 - u_-^2\right) \tag{1}$$

and

$$\begin{aligned} u - u^3 + \partial_{xx}u &= u_+ - u_- - 1 - (u_+ - u_- - 1)^3 + u_+^3 - u_+ - u_-^3 + u_- \quad (2)\\ &= 3u_+^2u_- - 3u_+u_-^2 + 3u_+^2 + 3u_-^2 - 6u_+u_- - 3u_+ + 3u_- \quad (3)\\ &= 3(u_+ - u_-)(u_+ - 1)(u_- + 1). \quad (4) \end{aligned}$$

3. Evaluating at $x_0(t)$, where $u_+(x_0) = \tanh(\sqrt{2}x_0)$ and $u_-(x_0) = 0$, we get

$$\dot{x}_0 \simeq -6\sqrt{2}\mathrm{e}^{-2\sqrt{2}x_0}. \tag{5}$$

4. This equation is solved by

$$\mathrm{e}^{2\sqrt{2}x_0(0)} - \mathrm{e}^{2\sqrt{2}x_0(t)} = 24t, \tag{6}$$

which leads to

$$x_0(t) = \frac{1}{2\sqrt{2}}\log\left(\mathrm{e}^{2\sqrt{2}x_0(t)} - 24t\right). \tag{7}$$

The collapse time is

$$t_c = \frac{\mathrm{e}^{2\sqrt{2}x_0(0)} - 1}{24} \simeq \frac{\mathrm{e}^{2\sqrt{2}x_0(0)}}{24}. \tag{8}$$

2 Growth or collapse of a 2D circular domain

5. For a circular domain with radius r, $c = \dot{r}$ and $\kappa = 1/r$, so that

$$\dot{r} = c_* - \frac{D}{r}. \tag{9}$$

This dynamics admits a stationary point $r_* = D/c_*$. We see that $\dot{r} > 0$ for $r > r_*$ and $\dot{r} < 0$ for $r < r_*$: this stationary point is unstable. The dynamics can be rewritten

$$\dot{r} = c_*\left(1 - \frac{r_*}{r}\right). \tag{10}$$

6. Dividing by the r.h.s. and integrating leads to

$$\int_{r(t)}^{r(0)} \frac{r\mathrm{d}r}{r_* - r} = c_* t. \tag{11}$$

Using the relation given in the question, we get

$$r_* \log\left(\frac{r_* - r(t)}{r_* - r(0)}\right) - r(0) + r(t) = c_* t. \tag{12}$$

7. The collapse time is given $r(t_c) = 0$:

$$c_* t_c = -r_* \log\left(1 - \frac{r(0)}{r_*}\right) - r(0). \tag{13}$$

In the limit $c_* \to 0$, $r_* \to \infty$, expanding the logarithm leads to

$$t_c = \frac{r(0)^2}{2D}. \tag{14}$$

The collapse is much faster in two (or more) dimensions than in one.

8. The normal propagation at a velocity c of a front with slope $\partial_x h = \tan(\theta)$ is

$$\partial_t h = \frac{c}{\cos(\theta)}. \tag{15}$$

To order h^2, we obtain

$$\partial_t h = c\left[1 + \frac{1}{2}(\partial_x h)^2\right]. \tag{16}$$

To the lowest order in h, the slope is given by $\kappa = \partial_x x h$. Finally, the equation is

$$\partial_t h = c + \frac{c}{2}(\partial_x h)^2 + D\partial_{xx} h. \tag{17}$$

We note that setting $u = \partial_x h$ and derivating the equation above, we obtain

$$\partial_t u = cu\partial_x u + D\partial_{xx} u : \tag{18}$$

this is the Burgers equation.