ICFP – Soft Matter

Einstein viscosity

Anke Lindner, Vincent Démery

The presence of solid particles suspended in a solvent modifies the rheological properties of the solvent. In the simplest case, the particles are spherical, non-deformable, and the suspension is dilute, so that the hydrodynamic interactions are negligible. The viscosity of this suspension has been computed by Einstein in 1906.

Here we follow Einstein's derivation of the flow around a spherical particle of radius a placed at the origin in an incompressible Newtonian fluid with viscosity μ submitted to a simple shear, $\boldsymbol{U}^\infty(\boldsymbol{x}) = \dot\gamma y \hat{\boldsymbol{x}}$ [1]. Then we use this flow to compute the average stress in the fluid.

Technical note: questions requiring calculations are indicated with asterisks: no asterisk for less than three lines of calculations, one for less than 10 lines, and two for longer calculations.

1 Motion of the sphere, equation for the flow

1. Decompose the flow $\boldsymbol{U}^\infty$ in a rotational part $\boldsymbol{U}^\infty_\mathrm{r}$ and a strain part $\boldsymbol{U}^\infty_\mathrm{s}$; note that these parts should have an antisymmetric and a symmetric gradient, respectively. In the following we denote E_{ij} the symmetric part.

2. What is the motion of the sphere? Explain why the rotational part of the flow is unperturbed, while the strain part is perturbed by the sphere.

3. We denote $\boldsymbol{u}(\boldsymbol{x})$ the perturbation of the flow, and $p(\boldsymbol{x})$ the pressure field. What equations do they satisfy (we assume a low Reynolds number)? What are the boundary conditions?

2 Solution for the flow

4. Show that the pressure field is harmonic ($\nabla^2 p = 0$).

We look for harmonic solutions that decay to zero at infinity, and that can have a singularity at the origin ($r = 0$) since we look for a solution over $r > a$. We know that $1/r$ is harmonic, and so are its derivatives.

5. * Express the derivatives $\partial_i(1/r)$, $\partial_i\partial_j(1/r)$, $\partial_i\partial_j\partial_k(1/r)$.

6. How can the scalar pressure field be constructed from the tensor E_{ij} and the derivatives calculated above? Express the general form of the pressure field, up to an unknown coefficient.

7. * Show that $u_i = x_i p/(2\mu)$ solves the Stokes equation for any admissible pressure field (incompressibility will be enforced later). Show that an harmonic flow field can be added to this special solution; how can such flow field be constructed from the tensor E_{ij} and derivatives of $1/r$? Conclude that the general solution for the flow field is

$$u_i = \frac{\lambda_1}{2\mu}\frac{E_{jk}x_ix_jx_k}{r^5} + \lambda_2\frac{E_{ij}x_j}{r^3} + \lambda_3 E_{jk}\left(\frac{\delta_{ij}x_k + \delta_{ik}x_j + \delta_{jk}x_i}{r^5} - \frac{5x_ix_jx_k}{r^7}\right). \tag{1}$$

8. * Use the incompressibility condition and the boundary conditions to determine the coefficients λ_i.

3 Average stress in the fluid and viscosity

The stress $\boldsymbol{\sigma}(\boldsymbol{x})$ in the fluid is given by

$$\sigma_{ij} = 2\mu e_{ij} - p\delta_{ij}, \tag{2}$$

where $e_{ij} = E_{ij} + (\partial_i u_j + \partial_j u_i)/2$ is the rate of strain tensor.

9. Compute $\partial_k(\sigma_{ik} x_j)$, and then show that the average stress in the fluid can be written as a surface term. Discuss how this surface term is related to what is measured by a rheometer.

10. ** Compute the average stress in a sphere with radius $R \gg a$ (so that you can keep only the dominant terms in the flow). We provide the following integrals over the unit sphere:

$$\int_{\mathcal{S}} x_i x_j \mathrm{d}\boldsymbol{x} = \frac{4\pi}{3} \delta_{ij}, \tag{3}$$

$$\int_{\mathcal{S}} x_i x_j x_k x_l \mathrm{d}\boldsymbol{x} = \frac{4\pi}{15} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{4}$$

11. * Show that the average rate of strain in the sphere of radius R is given by

$$\bar{e}_{ij} = \left(1 - \frac{a^3}{R^3}\right) E_{ij}. \tag{5}$$

12. Sum over the spheres of the suspension, show that the viscosity of the suspension is given by

$$\mu_E = \mu \left(1 + \frac{5}{2}\phi\right), \tag{6}$$

where ϕ is the volumic fraction of the spheres. This is the Einstein viscosity.

References

[1] Élisabeth Guazzelli, Jeffrey F. Morris, and Sylvie Pic. *A Physical Introduction to Suspension Dynamics.* Cambridge Texts in Applied Mathematics. Cambridge University Press, 2011.