ICFP – Soft Matter

Einstein viscosity – Solution

Anke Lindner, Vincent Démery

We mostly follow Ref. [1], in particular for the calculation of the flow. Another useful reference is Ref. [2].

1 Flow

1. We impose a shear flow at infinity: $\boldsymbol{u}^\infty(\boldsymbol{x}) = \dot\gamma y\hat{\boldsymbol{x}}$. We use Stokes equations, hence everything (flow and motion of the sphere) should be linear in the imposed flow. We decompose $\boldsymbol{u}^\infty = \boldsymbol{u}^\infty_{\text{rot}} + \boldsymbol{u}^\infty_{\text{strain}}$, where

$$\boldsymbol{u}^\infty_{\text{rot}}(\boldsymbol{x}) = \frac{\dot\gamma}{2}(y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}}) \tag{1}$$

is a pure rotation, and

$$\boldsymbol{u}^\infty_{\text{strain}}(\boldsymbol{x}) = \frac{\dot\gamma}{2}(y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}}) \tag{2}$$

is a pure strain.

2. The rotation enforces a rotation of the sphere with the rate $-\dot\gamma/2$ around $\hat{\boldsymbol{z}}$; this flow is not disturbed. On the other hand

$$\boldsymbol{u}^\infty_{\text{strain}}(\boldsymbol{x}) = \frac{\dot\gamma}{2}(y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}}) \tag{3}$$

is a pure strain. As the sphere is rigid, this flow must be disturbed to ensure the boundary condition $\boldsymbol{u}(|\boldsymbol{x}| = a) = 0$.

3. We just have to study the disturbance created by the strain; we denote $\boldsymbol{u}^\infty_{\text{strain}} = \boldsymbol{u}^\infty$, and $\boldsymbol{u}$ the disturbance. We have

$$u_i^\infty(\boldsymbol{x}) = E_{ij}x_j, \tag{4}$$

with $E_{ii} = 0$, and the boundary condition is $u_i(r = a) = -E_{ij}x_j$, and $u_i(r \to \infty) \to 0$. The disturbance should satisfy the Stokes equations

$$\partial_i u_i = 0, \tag{5}$$

$$\mu\nabla^2 u_i = \partial_i p, \tag{6}$$

where p is the pressure and μ is the viscosity.

2 Solution for the flow

4. Combining equations (5,6), we find $\partial_i\partial_i p = \mu\nabla^2\partial_i u_i = 0$: p is harmonic.

5. To compute the derivatives of $1/r$, we use $\partial_i r = x_i/r$ and $\partial_i x_j = \delta_{ij}$. We get:

$$\partial_i\left(\frac{1}{r}\right) \propto \frac{x_i}{r^3}, \tag{7}$$

$$\partial_i\partial_j\left(\frac{1}{r}\right) \propto \frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}, \tag{8}$$

$$\partial_i\partial_j\partial_k\left(\frac{1}{r}\right) \propto \frac{\delta_{ij}x_k + \delta_{ik}x_j + \delta_{jk}x_i}{r^5} - \frac{5x_ix_jx_k}{r^7}. \tag{9}$$

6. The only way to form a scalar with the tensor E_{ij} and these quantities is to have

$$p \propto E_{ij}\partial_i\partial_j\left(\frac{1}{r}\right) \propto E_{ij}\frac{x_i x_j}{r^5}, \tag{10}$$

where we have used that $E_{ii} = 0$. Hence

$$p = \lambda_1 E_{ij}\frac{x_i x_j}{r^5}. \tag{11}$$

7. We then have to determine the flow $\boldsymbol{u}$: it can be decomposed in a special solution to (6) and a harmonic part. The special solution can be written $u_i = x_i p/(2\mu)$, indeed:

$$\partial_j\partial_j(x_i p) = (\partial_j\partial_j x_i)p + 2(\partial_j x_i)(\partial_j p) + x_i\partial_j\partial_j p = 2\delta_{ij}\partial_j p = 2\partial_i p. \tag{12}$$

The harmonic solution has to be formed from the derivatives of $1/r$ above, leading to

$$u_i = \frac{\lambda_1}{2\mu}\frac{E_{jk}x_i x_j x_k}{r^5} + \lambda_2\frac{E_{ij}x_j}{r^3} + \lambda_3 E_{jk}\left(\frac{\delta_{ij}x_k + \delta_{ik}x_j + \delta_{jk}x_i}{r^5} - \frac{5x_i x_j x_k}{r^7}\right). \tag{13}$$

8. In order to enforce incompressibility, we have to compute the divergence of the three terms. The divergence of the first term can be computed and is zero. The last term is $E_{jk}\partial_i\partial_j\partial_k(1/r)$ and since $1/r$ is harmonic the divergence of this term is zero. The divergence of the second term is not zero, which sets $\lambda_2 = 0$.

The boundary condition $u_i(r = a) = -E_{ij}x_j$ leads to $\frac{\lambda_1}{2\mu a^5} = \frac{5\lambda_3}{a^7}$ and $2\lambda_3/a^5 = -1$, hence

$$p = -5\mu a^3\frac{E_{ij}x_i x_j}{r^5}, \tag{14}$$

$$u_i = -\frac{5}{2}\frac{a^3}{r^5}E_{jk}x_i x_j x_k\left(1 - \frac{a^2}{r^2}\right) - \frac{a^5}{r^5}E_{ij}x_j. \tag{15}$$

3 Average stress in the fluid and viscosity

9. We have that $\partial_k(\sigma_{ik}x_j) = \sigma_{ij} + (\partial_k\sigma_{ik})x_j = \sigma_{ij}$, using that $\partial_k\sigma_{ik} = 0$. Hence, we can write

$$\bar{\sigma}_{ij} = \frac{1}{V}\int_{\mathcal{V}}\sigma_{ij}\mathrm{d}\boldsymbol{x} = \frac{1}{V}\int_{\mathcal{V}}\partial_k(\sigma_{ik}x_j)\mathrm{d}\boldsymbol{x} = \frac{1}{V}\int_{\partial\mathcal{V}}\sigma_{ik}x_j n_k\mathrm{d}\boldsymbol{x}, \tag{16}$$

where n_k is a unit vector pointing towards the outside of the volume $\mathcal{V}$. This quantity is actually what is measured by a rheometer (which measures, for instance, the torque on the top plate).

10. Keeping only the dominant terms, we get

$$p = -5\mu a^3\frac{E_{ij}x_i x_j}{r^5}, \tag{17}$$

$$u_i = -\frac{5}{2}\frac{a^3}{r^5}E_{jk}x_i x_j x_k. \tag{18}$$

The stress disturbance is

$$\delta\sigma_{ij} = \mu(\partial_i u_j + \partial_j u_i) - p\delta_{ij} = 5\mu a^3\left(-\frac{E_{ik}x_j x_k + E_{jk}x_i x_k}{r^5} + 5E_{kl}\frac{x_i x_j x_k x_l}{r^7}\right). \tag{19}$$

Integrating over the sphere of radius R, $\mathcal{S}_R$, using that $n_i = x_i/R$, we get

$$\delta\bar{\sigma}_{ij} = \frac{5\mu a^3}{RV}\int_{\mathcal{S}_R}\left(-E_{ik}\frac{x_j x_k}{R^3} + 4E_{kl}\frac{x_i x_j x_k x_l}{R^5}\right) \tag{20}$$

$$= \frac{5\mu a^3}{V}\int_{\mathcal{S}_1}(-E_{ik}x_j x_k + 4E_{kl}x_i x_j x_k x_l) \tag{21}$$

$$= \frac{5\mu a^3}{V}\left[-E_{ik}\frac{4\pi}{3}\delta_{jk} + 4E_{kl}\frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right] \tag{22}$$

$$= \frac{4\pi\mu a^3}{V}E_{ij}. \tag{23}$$

Note that this is the additionnal stress due to the disturbance $\boldsymbol{u}$. The total average stress is

$$\bar{\sigma}_{ij} = 2\mu E_{ij}\left(1+\frac{2\pi a^3}{V}\right) = 2\mu E_{ij}\left(1+\frac{3v}{2V}\right), \tag{24}$$

where $v = (4/3)\pi a^3$ is the volume of a small sphere.

11. The average strain disturbance is

$$\begin{aligned}
\delta\bar{e}_{ij} &= \frac{1}{2V}\int_{\mathcal{V}}(\partial_i u_j + \partial_j u_i) && (25)\\
&= \frac{1}{2VR}\int_{\mathcal{S}_R}(x_i u_j + x_j u_i) && (26)\\
&= -\frac{5a^3}{2V}\int_{\mathcal{S}_1} E_{kl}x_i x_j x_k x_l && (27)\\
&= -\frac{v}{V}E_{ij}. && (28)
\end{aligned}$$

The average strain is thus

$$\bar{e}_{ij} = \left(1-\frac{v}{V}\right)E_{ij}. \tag{29}$$

12. Summing the response over all the spheres and using the volume fraction ϕ, we get

$$\begin{aligned}
\bar{\sigma}_{ij} &= 2\mu E_{ij}\left(1+\frac{3}{2}\phi\right), && (30)\\
\bar{e}_{ij} &= (1-\phi)\,E_{ij}. && (31)
\end{aligned}$$

For small volume fraction, inverting the second equation gives $E_{ij} = (1+\phi)\bar{e}_{ij}$ and

$$\bar{\sigma}_{ij} = 2\mu\bar{e}_{ij}\left(1+\frac{5}{2}\phi\right) = 2\mu_E\bar{e}_{ij}, \tag{32}$$

where

$$\mu_E = \mu\left(1+\frac{5}{2}\phi\right) \tag{33}$$

is the Einstein viscosity.

3.1 Alternative calculation of the viscosity

This is the calculation given in Ref. [1].

The average stress in the fluid is

$$\bar{\sigma}_{ij} = \frac{1}{V}\int_{\mathcal{V}}\sigma_{ij}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 2\mu\bar{e}_{ij} - \bar{p}\delta_{ij} + \frac{1}{V}\int_{\mathcal{V}}\left[\sigma_{ij}(\boldsymbol{x}) - 2\mu e_{ij}(\boldsymbol{x}) + p(\boldsymbol{x})\delta_{ij}\right]\mathrm{d}\boldsymbol{x}; \tag{34}$$

note that the integrand is non zero over the particles only. The last term in the integrand and $\bar{p}$ should vanish by symmetry. Noting that $\sigma_{ij} = \partial_k(\sigma_{ik}x_j)$, we can transform the integral in a surface integral

$$\bar{\sigma}_{ij} = 2\mu E_{ij} + \frac{1}{V}\int_{\mathcal{S}_R}\left[\sigma_{ik}x_j n_k - \mu(n_i u_j + n_j u_i)\right]\mathrm{d}\boldsymbol{x}, \tag{35}$$

where the integral is performed over the sphere of radius R. Choosing $R\to\infty$, we just have to care about the dominant component of the flow,

$$u_i^{\mathrm{dom}} = -\frac{5}{2}\frac{a^3}{r^5}E_{jk}x_i x_j x_k, \tag{36}$$

it is associated to a strain rate

$$e_{ij}^{\mathrm{dom}} = -\frac{5a^3}{2}\left[\frac{E_{kl}x_k x_l}{r^5}\left(\delta_{ij} - \frac{5x_i x_j}{r^2}\right) + \frac{E_{ik}x_j x_k + E_{jk}x_i x_k}{r^5}\right]. \tag{37}$$

With the stress $\sigma_{ij}^{\mathrm{dom}} = 2\mu e_{ij}^{\mathrm{dom}} - p\delta_{ij}$, the surface integral reads (using the surface integrals given in App. A)

$$\int_{\mathcal{S}_R} \left[\sigma_{ik} x_j n_k - \mu(n_i u_j + n_j u_i)\right] \mathrm{d}\boldsymbol{x} = \frac{20\pi}{3} \mu a^3 E_{ij}. \tag{38}$$

Summing over the N particles in the suspension, and using the volume fraction $\phi = \frac{4\pi a^3 N}{3V}$, we get

$$\bar{\sigma}_{ij} = 2\mu \left(1 + \frac{5}{2}\phi\right) E_{ij}, \tag{39}$$

where the Einstein viscosity appears:

$$\mu_E = \mu \left(1 + \frac{5}{2}\phi\right). \tag{40}$$

A Surface integrals of polynomials

Using spherical coordinates, we can compute the following integrals over the unit sphere:

$$\int_{\mathcal{S}} x_i x_j \mathrm{d}\boldsymbol{x} = \frac{4\pi}{3} \delta_{ij}, \tag{41}$$

$$\int_{\mathcal{S}} x_i x_j x_k x_l \mathrm{d}\boldsymbol{x} = \frac{4\pi}{15} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{42}$$

References

[1] Élisabeth Guazzelli, Jeffrey F. Morris, and Sylvie Pic. *A Physical Introduction to Suspension Dynamics.* Cambridge Texts in Applied Mathematics. Cambridge University Press, 2011.

[2] L. D. Landau and E. M. Lifshitz. *Fluid Mechanics, Second Edition: Volume 6.* Course of theoretical physics / by L. D. Landau and E. M. Lifshitz, Vol. 6. Butterworth-Heinemann, 2 edition, jan 1987.