ICFP – Soft Matter

Shear-thickening of dense suspensions – Solution

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1 Single parameter rheology

1. As Φ is a decreasing function, it can be inverted: $I_v = \Phi^{-1}(\phi)$, which we denote $I_v(\phi)$. Then the pressure is given by $P = \eta_f\dot\gamma/I_v(\phi)$ and the shear stress is $\tau = P\mu(I_v(\phi)) = \eta_f\dot\gamma\mu(I_v(\phi))/I_v(\phi)$. We thus get $\eta_n(\phi) = 1/I_v(\phi)$ and $\eta_s = \mu(I_v(\phi))/I_v(\phi)$. The fluid does not shear-thicken.

2. In a Newtonian fluid, the pressure, which is a scalar, should depend on the scalar invariant of the strain tensor. The shear rate $\dot\gamma$ is not such an invariant. For instance, inverting the shear $\dot\gamma \to -\dot\gamma$ does not invert the pressure.

3. The data is compatible with

$$\Phi(I_v) = \frac{\phi_m}{1 + I_v^{1/2}}, \tag{1}$$

where ϕ_m is the maximal volume fraction under shear, $\phi_m \simeq 0.585$, different from the random close packing density $\phi_0 \simeq 0.64$. Inverting this relation leads to $I_v \sim (\phi_m - \phi)^2$ as $\phi \to \phi_m$, and $\mu(I_v) \to \mu_1$ as $I_v \to 0$, hence the normal and shear viscosities diverge as ϕ approaches ϕ_m.

4. For the Einstein viscosity to be recovered in the limit $\phi \to 0$ we should have

$$\eta_s(\phi) = 1 + \frac{5}{2}\phi. \tag{2}$$

In the low density regime, $\phi \simeq \phi_m/I_v^{1/2}$, hence

$$\eta_s(\phi) = 1 + \frac{5}{2}\phi = 1 + \frac{5}{2}\frac{\phi_m}{I_v^{1/2}} = I_v^{-1}\left(I_v + \frac{5}{2}\phi_m I_v^{1/2}\right) = \frac{\mu(I_v)}{I_v}, \tag{3}$$

leading to

$$\mu(I_v) \simeq I_v + \frac{5}{2}\phi_m I_v^{1/2}. \tag{4}$$

This is indeed the observed scaling for $I_v \gg 1$ [1]. In this regime, we see that $P = \eta_f\dot\gamma/I_v \sim \eta_f\dot\gamma(\phi/\phi_m)^2$: the pressure goes to zero faster than the shear stress. The quadratic dependence points to an effect of the interaction between the suspended particles, which is not taken into account in the Einstein calculation.

2 Discontinuous shear-thickening with repulsive interactions

5. From the equations for p and $\phi_J(p)$, we get (discarding the constant λ)

$$\dot\gamma = p[\phi_m + (\phi_0 - \phi_m)\mathrm{e}^{-p} - \phi]^2. \tag{5}$$

As the term in brackets is a decreasing function of p, the suspension generally shear-thickens. We now analyze this relation in more detail.

For $\phi > \phi_0$, the suspension is jammed for any pressure.

For $\phi_m < \phi < \phi_0$, the suspension can flow at low pressure, with $\dot\gamma \simeq p(\phi_0 - \phi)^2$, but jams at

$$p_J = \log\left(\frac{\phi_0 - \phi_m}{\phi - \phi_m}\right). \tag{6}$$

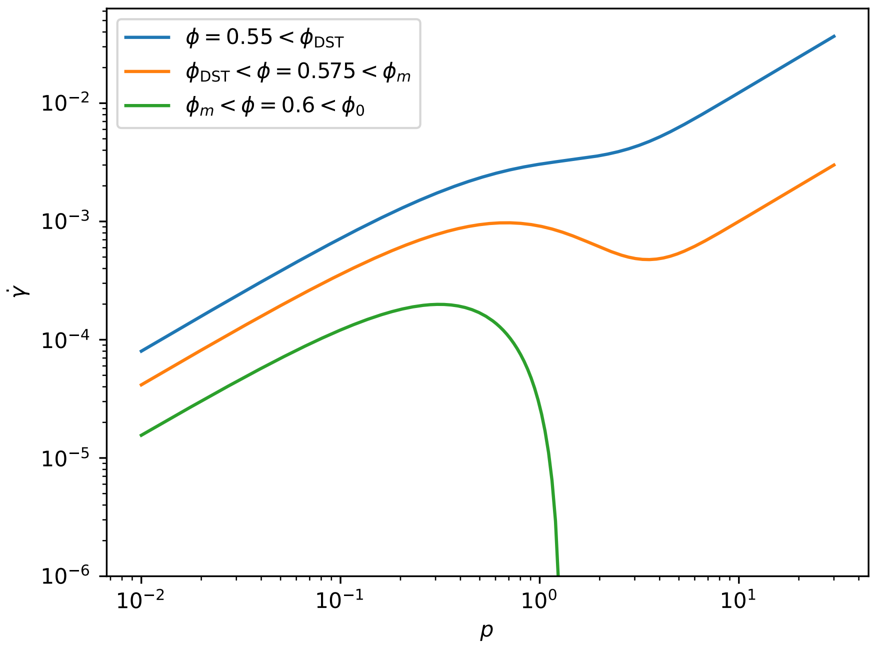

Figure 1: Strain rate as the function of the pressure for different values of the volume fraction, from Eq. (5) for $\phi_0 = 0.64$, $\phi_m = 0.585$, leading to $\phi_{\rm DST} \simeq 0.56$.

For $\phi < \phi_m$, the suspension flows at all pressures, with $\dot{\gamma} \simeq p(\phi_0 - \phi)^2$ at small pressures and $\dot{\gamma} \simeq p(\phi_m - \phi)^2$ at large pressures. The shear thickening is discontinuous if $\partial\dot{\gamma}/\partial p = 0$. Computing the derivative, we find that

$$\frac{\partial \dot{\gamma}}{\partial p} = \left[\phi_m + (\phi_0 - \phi_m)\mathrm{e}^{-p} - \phi\right] \times \left[\phi_m - \phi + (1 - 2p)(\phi_0 - \phi_m)\mathrm{e}^{-p}\right]. \tag{7}$$

For the derivative to vanish, we should have

$$g(p) = (2p - 1)\mathrm{e}^{-p} = \frac{\phi_m - \phi}{\phi_0 - \phi_m}, \tag{8}$$

meaning that the maximum of g, g^*, should be larger than the right hand side. We have $g'(p) = (3 - 2p)\mathrm{e}^{-p}$, hence the maximum is reached at $p^* = 3/2$, and $g^* = g(p^*) = 2\mathrm{e}^{-3/2} \simeq 0.446$. The derivative may vanish, indicating a discontinuous shear thickening, for

$$\phi > \phi_{\rm DST} = \phi_m - 2\mathrm{e}^{-3/2}(\phi_0 - \phi_m). \tag{9}$$

Flow curves in the three flowing situations are presented in Fig. 1.

References

[1] François Boyer, Élisabeth Guazzelli, and Olivier Pouliquen. Unifying Suspension and Granular Rheology. *Phys. Rev. Lett.*, 107(18):188301, Oct 2011.