ICFP – Soft Matter

Elasto-plastic models for the flow of amorphous solids

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Amorphous solids, from metallic and colloidal glasses to emulsions and foams, have a disordered structure, like liquids, but can resist shear, like solids. However, they flow if they are sheared strong enough: they have a yield stress σ_y. Above the yield stress, their flow curve follows the Herschel-Bulkley law, $\dot\gamma\sim(\sigma-\sigma_y)^\alpha$, with $\alpha\simeq 2$.

Microscopically, these materials are made of a dense assembly of particles. When they are sheared, they respond elastically until a local yield stress is reached somewhere in a sample and a shear transformation occurs, whereby, for instance, the local network of neighbors is modified. This local rearrangement induces a stress redistribution in the sample, which can take other points above their yield stress. This behavior is captured by the elastoplastic models, which are mesoscopic models operating at the scale of the shear transformation zones [1].

In this tutorial, we present a very basic, yet rich enough elastoplastic model, and start with an equation for the evolution of the probability $P((\sigma_i)_{1\leq i\leq N})$ of finding each site i in a state of stress σ_i. First, we derive a mean-field model, the Hébraud-Lequeux model, that can be treated analytically [2]. Second, we investigate the rheology of the Hébraud-Lequeux model.

Technical note: questions requiring calculations are indicated with asterisks: no asterisk for less than three lines of calculations, one for less than 10 lines, and two for longer calculations.

1 Mean-field treatment of an elastoplastic model

We consider a system with N sites carying a scalar shear stress σ_i. The system, with elastic constant μ, is submitted to a shear rate $\dot\gamma$. Without plasticity, the evolution of the shear stress of each site is given by $\dot\sigma_i=\mu\dot\gamma$. We introduce a plasticity rule: when $|\sigma_i|>\sigma_\mathrm{c}$, the site i yields with a rate τ^{-1}. When the site i yields, its stress drops to 0 and the stress at another site j increases by G_{ij}, where G_{ij} describes the stress redistribution.

1. Explain the origin of the different terms entering the following master equation for the probability $P((\sigma_i)_{1\leq i\leq N})$ of finding the system in a state of stress (σ_i),

$$\partial_t P=-\mu\dot\gamma\sum_i\partial_{\sigma_i}P+\tau^{-1}\sum_i\left[-\theta(|\sigma_i|-\sigma_\mathrm{c})P+\delta(\sigma_i)\int_{|\sigma_i'|>\sigma_\mathrm{c}}P(\sigma_i',(\sigma_j-G_{ij})_{j\neq i},t)\mathrm{d}\sigma_i'\right],\tag{1}$$

where θ is the Heaviside function.

2. * Integrate the equation for $P((\sigma_i)_{1\leq i\leq N})$ over all the stresses but σ_1 in order to obtain an equation for the maginal probability $P_1(\sigma)=\int P(\sigma,(\sigma_i)_{i\neq 1})\prod_{i\neq 1}\mathrm{d}\sigma_i$. You can introduce $P_{1i}(\sigma,\sigma')=\int P(\sigma,\sigma',(\sigma_j)_{j\neq 1,i})\prod_{j\neq 1,i}\mathrm{d}\sigma_j$.

3. Can we get a closed equation for $P_1(\sigma)$? What is the type of system of equations that we can obtain?

4. Perform a mean-field approximation and assume a homogeneous system to obtain the following closed equation for $P_1(\sigma)$:

$$\partial_t P_1(\sigma)=-\mu\dot\gamma\partial_\sigma P_1(\sigma)+\tau^{-1}\left(-\theta(|\sigma|-\sigma_\mathrm{c})P_1(\sigma)+\Gamma\delta(\sigma)+\Gamma\sum_{i\neq 1}\left[P_1(\sigma-G_{i1})-P_1(\sigma)\right]\right),\tag{2}$$

where $\Gamma=\int_{|\sigma|>\sigma_\mathrm{c}}P_1(\sigma)\mathrm{d}\sigma$.

5. Argue that, if the sign of G_{i1} has an even distribution (it can take the values g and $-g$ with equal probability), the sum over i can be replaced by $\alpha\Gamma\partial_\sigma^2P_1(\sigma)$: this is the Hébraud-Lequeux model.

2 Rheology of the Hébraud-Lequeux model

The Hébraud-Lequeux model thus reads

$$\partial_t P(\sigma) = -\mu\dot{\gamma}\partial_\sigma P(\sigma) + \frac{\alpha\Gamma}{\tau}\partial_\sigma^2 P(\sigma) - \tau^{-1}\theta(|\sigma| - \sigma_\mathrm{c})P(\sigma) + \frac{\Gamma}{\tau}\delta(\sigma), \tag{3}$$

again with $\Gamma = \int_{|\sigma|>\sigma_\mathrm{c}} P(\sigma)\mathrm{d}\sigma$.

6. What are the stationary solutions satisfying $\Gamma = 0$? What kind of states do they correspond to?

7. * When $\dot{\gamma} = 0$, find the stationary solution with $\Gamma > 0$. Show that there is a transition for some value of α, α_c. What is the behavior of the stress in such state?

8. * In the general case, write down the equations to be solved and try to guess the response of the system to a small shear rate, above and below α_c.

References

[1] Alexandre Nicolas, Ezequiel E. Ferrero, Kirsten Martens, and Jean-Louis Barrat. Deformation and flow of amorphous solids: Insights from elastoplastic models. *Rev. Mod. Phys.*, 90(4):045006, Dec 2018.

[2] P. Hébraud and F. Lequeux. Mode-Coupling Theory for the Pasty Rheology of Soft Glassy Materials. *Phys. Rev. Lett.*, 81(14):2934–2937, Oct 1998.