ICFP – Soft Matter

Elasto-plastic models for the flow of amorphous solids – Solution

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1 Mean-field treatment of an elastoplastic model

1. In the proposed equation,

$$\partial_t P = -\mu\dot\gamma \sum_i \partial_{\sigma_i} P + \tau^{-1} \sum_i \left[-\theta(|\sigma_i| - \sigma_\mathrm{c})P + \delta(\sigma_i) \int_{|\sigma_i|>\sigma_\mathrm{c}} P(\sigma_i, (\sigma_j - G_{ij})_{j\neq i}, t)\mathrm{d}\sigma_i \right], \tag{1}$$

on the right:

- the first term on the right represents the advection of the stress distribution by $\mu\dot\gamma$,
- the second term represents the yield with rate τ^{-1} for each site above the yield stress σ_c,
- the third term represents the appearance of states with no stress at site i after yielding.

One can check the consistency of this equation by showing that the integral over all the stresses is zero on both sides.

2. The marginal probability is defined by

$$P_1(\sigma, t) = \int P(\sigma, (\sigma_i)_{i>1}, t) \prod_{i>1} \mathrm{d}\sigma_i. \tag{2}$$

We also introduce the two-point stress probability distribution

$$P_{1i}(\sigma, \sigma', t) = \int P(\sigma, \sigma', (\sigma_j)_{j\neq 1,i}, t) \prod_{j\neq 1,i} \mathrm{d}\sigma_j. \tag{3}$$

With these definitions, integrating Eq. (1) over all the stresses but $\sigma_1 = \sigma$, we get

$$\begin{aligned}\partial_t P_1(\sigma, t) = -\mu\dot\gamma\partial_\sigma P_1(\sigma, t) + \tau^{-1} \left[-\theta(\sigma - \sigma_\mathrm{c})P_1(\sigma, t) + \delta(\sigma) \int_{|\sigma'|>\sigma_\mathrm{c}} P_1(\sigma', t)\mathrm{d}\sigma' \right] \\ + \tau^{-1} \sum_{i>1} \left[-\int_{|\sigma'|>\sigma_\mathrm{c}} P_{1i}(\sigma, \sigma', t)\mathrm{d}\sigma' + \int_{|\sigma'|>\sigma_\mathrm{c}} P_{1i}(\sigma - G_{1i}, \sigma', t)\mathrm{d}\sigma' \right].\end{aligned} \tag{4}$$

The first bracketted term corresponds to the term $i = 1$ of the sum and represents the yielding of site 1, while the second represents the yielding of all the other sites.

3. We cannot obtain a closed equation for $P_1(\sigma)$, this is the standard BBGKY hierarchy.

4. We assume a decoupling of the stress between the different sites (this is the starting point of Ref. [1]); moreover, we assume that the stress probability distribution is the same for all the sites:

$$P_{1i}(\sigma, \sigma', t) = P_1(\sigma, t) \times P_i(\sigma', t) = P_1(\sigma, t) \times P_1(\sigma', t); \tag{5}$$

this is our only assumption. With this assumtion, we get

$$\partial_t P_1(\sigma, t) = -\mu\dot\gamma\partial_\sigma P_1(\sigma, t) - \tau^{-1}\theta(\sigma - \sigma_\mathrm{c})P_1(\sigma, t) + \Gamma(t)\delta(\sigma) + \Gamma(t) \sum_{i>1} \left[P_1(\sigma - G_{1i}, t) - P_1(\sigma, t)\right], \tag{6}$$

where we have introduced the plastic activity

$$\Gamma(t) = \int_{|\sigma'|>\sigma_c} P_1(\sigma,t)\mathrm{d}\sigma. \tag{7}$$

5. We can Taylor expand the distribution around σ:

$$P_1(\sigma - G_{i1}) = \sum_{n\geq 0} \frac{G_{i1}^n}{n!}\partial_\sigma^n P_1(\sigma). \tag{8}$$

The sum over i becomes

$$\sum_{i>1}[P_1(\sigma - G_{1i},t) - P_1(\sigma,t)] = \sum_{n\geq 1}\frac{1}{n!}\partial_\sigma^n P_1(\sigma)\sum_i G_{i1}^n. \tag{9}$$

Since the terms G_{i1} are evenly distributed, $\sum_i G_{i1} = 0$. The lowest order non-zero term thus involves the second derivative of the distribution, $\partial_\sigma^2 P_1(\sigma)$, and we retain only this term in the following. This is the "Kramers-Moyal expansion".

2 Rheology of the Hébraud-Lequeux model

6. The distributions where $\Gamma = 0$ correspond to distributions where $P(\sigma) = 0$ for $|\sigma| \geq \sigma_c$; they exist only without shear strain, $\dot\gamma = 0$. In such configurations, there is no yielding and nothing happens: the system is solid.

7. In the stationary state when $\dot\gamma = 0$, the stationary distribution follows

$$\alpha\Gamma\partial_\sigma^2 P(\sigma) - \theta(|\sigma| - \sigma_c)P(\sigma) + \Gamma\delta(\sigma) = 0. \tag{10}$$

The distribution is piecewise linear for $|\sigma| < \sigma_c$, with $P'(0^+) - P'(0^-) = -\alpha^{-1}$, hence by symmetry $P(\sigma) = A - |\sigma|/(2\alpha)$. For $|\sigma| > \sigma_c$, $P(\sigma) = B\exp\left(-(|\sigma| - \sigma_c)/\sqrt{\alpha\Gamma}\right)$. Continuity sets $A = B + \sigma_c/(2\alpha)$ and continuity of the derivative sets $1/(2\alpha) = B/\sqrt{\alpha\Gamma}$, hence $B = \sqrt{\Gamma/\alpha}/2$.

The normalization of the distribution reads

$$1 = \int P(\sigma)\mathrm{d}\sigma = \Gamma + \sigma_c\left(\sqrt{\frac{\Gamma}{\alpha}} + \frac{\sigma_c}{2\alpha}\right). \tag{11}$$

There is a solution for Γ only if

$$\alpha \geq \alpha_c = \frac{\sigma_c^2}{2}. \tag{12}$$

As a conclusion, for $\alpha < \alpha_c$, all the possible solutions correspond to a stress distribution where all the stresses are below the threshold and no yielding occurs, the system is solid. For $\alpha > \alpha_c$ there is also a stationnary distribution where some sites are above the threshold and yield, leading to a dynamic stress evolution in the sample. This looks like a liquid. We may conjecture that a yield stress exists for $\alpha < \alpha_c$, while there is none for $\alpha > \alpha_c$.

8. When $\dot\gamma \neq 0$ we have to solve

$$0 = -\tau\mu\dot\gamma\partial_\sigma P(\sigma) + \alpha\Gamma\partial_\sigma^2 P(\sigma) - \theta(|\sigma| - \sigma_c)P(\sigma) + \Gamma\delta(\sigma), \tag{13}$$

Again we divide the system in intervals. For $|\sigma| < \sigma_c$, the probability reads

$$P(\sigma) = \begin{cases} A_2 + B_2\mathrm{e}^{a_2\sigma} & \text{for} \quad -\sigma_c < \sigma < 0, \\ A_3 + B_3\mathrm{e}^{a_2\sigma} & \text{for} \quad 0 < \sigma < \sigma_c, \end{cases} \tag{14}$$

where

$$a_2 = \frac{\tau\mu\dot\gamma}{\alpha\Gamma}. \tag{15}$$

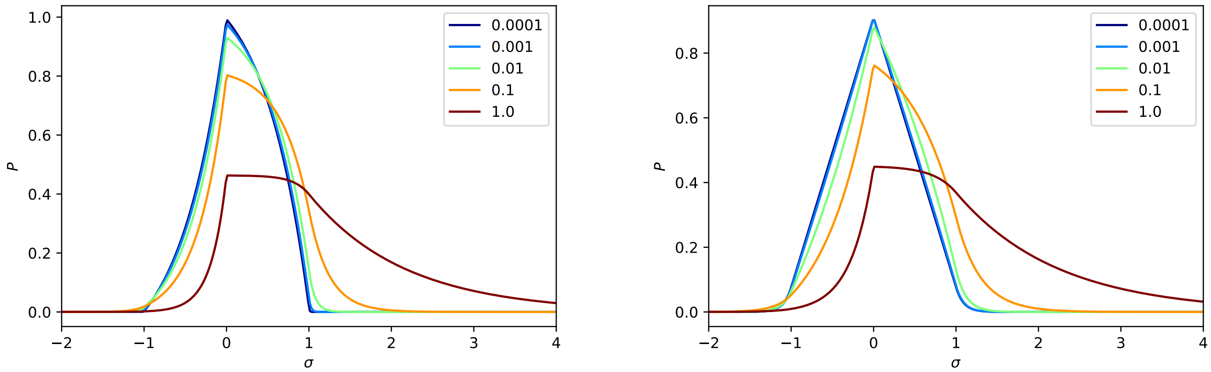

Figure 1: Stationnary stress distribution in the Hébraud-Lequeux model (solutions to Eq. (13)) for different values of $\dot{\gamma}$ and $\alpha/\sigma_c^2 = 0.4$ (left) and 0.6 (right).

For $|\sigma| > \sigma_c$ we need to solve $\alpha\Gamma x^2 - \tau\mu\dot{\gamma}x - 1 = 0$, which has two solutions

$$a_1 = \frac{\tau\mu\dot{\gamma}}{2\alpha\Gamma} + \sqrt{\left(\frac{\tau\mu\dot{\gamma}}{2\alpha\Gamma}\right)^2 + \frac{1}{\alpha\Gamma}}, \tag{16}$$

$$a_4 = \frac{\tau\mu\dot{\gamma}}{2\alpha\Gamma} - \sqrt{\left(\frac{\tau\mu\dot{\gamma}}{2\alpha\Gamma}\right)^2 + \frac{1}{\alpha\Gamma}}. \tag{17}$$

The distribution reads

$$P(\sigma) = \begin{cases} A_1 e^{a_1(\sigma+\sigma_c)} & \text{for} \quad \sigma < -\sigma_c, \\ A_4 e^{a_4(\sigma-\sigma_c)} & \text{for} \quad \sigma > \sigma_c. \end{cases} \tag{18}$$

The boundary conditions read

$$A_1 = A_2 + e^{-a_2\sigma_c} B_2, \tag{19}$$

$$a_1 A_1 = a_2 e^{-a_2\sigma_c} B_2, \tag{20}$$

$$A_2 + B_2 = A_3 + B_3, \tag{21}$$

$$a_2 B_2 = a_2 B_3 + \frac{1}{\alpha}, \tag{22}$$

$$A_4 = A_3 + e^{-a_2\sigma_c} B_3, \tag{23}$$

$$a_4 A_4 = a_2 e^{a_2\sigma_c} B_3. \tag{24}$$

From these equations, one can determine all the coefficients. Then, the normalisation imposes a constraint on Γ, which takes the form of a non-linear equation. These equations are solved in Appendix A.

We can try to guess qualitatively the behavior of the system:

- For $\alpha < \alpha_c$, the activity Γ goes to 0 as $\dot{\gamma} \to 0$ and the limiting distribution has a finite average stress $\langle\sigma\rangle$, which is the yield stress. The yield stress should go to zero in the limit $\alpha \to \alpha_c^-$.
- For $\alpha > \alpha_c$, as $\dot{\gamma} \to 0$ the distribution is only slightly perturbed compared to the distribution for $\dot{\gamma} = 0$, hence the average stress goes to zero: there is no yield stress. The activity Γ goes to a finite value in this limit.

This is what is found by plotting numerically the stress distribution (Fig. 1).

A Full solution of the Hébraud-Lequeux model

For completeness, the coefficients are given by

$$A_1 = \frac{a_2 - a_4(1-q^{-1})}{\alpha\left[a_2(a_1 q - a_4 q^{-1}) - a_1 a_4 (q - q^{-1})\right]}, \tag{25}$$

$$A_4 = \frac{a_2 + a_1(q-1)}{\alpha\left[a_2(a_1 q - a_4 q^{-1}) - a_1 a_4 (q - q^{-1})\right]}, \tag{26}$$

$$A_2 = \left(1 - \frac{a_1}{a_2}\right) A_1, \tag{27}$$

$$B_2 = \frac{a_1}{a_2} q A_1, \tag{28}$$

$$A_3 = \left(1 - \frac{a_4}{a_2}\right) A_4, \tag{29}$$

$$B_3 = \frac{a_4}{a_2} q^{-1} A_4. \tag{30}$$

where we have defined $q = \mathrm{e}^{a_2 \sigma_\mathrm{c}}$, and the normalisation condition reads

$$1 = \Gamma + \left[\sigma_\mathrm{c}\left(1 - \frac{a_1}{a_2}\right) + \frac{a_1}{a_2^2}(q-1)\right] A_1 + \left[\sigma_\mathrm{c}\left(1 - \frac{a_4}{a_2}\right) + \frac{a_4}{a_2^2}(1-q^{-1})\right] A_4. \tag{31}$$

This equation for Γ can be solved numerically or analytically in limiting cases, such as $\dot{\gamma} \to 0$.

Once it has been solved, one can compute the average stress:

$$\langle\sigma\rangle = \int \sigma P(\sigma) \mathrm{d}\sigma \tag{32}$$

$$= -\frac{a_1\sigma_\mathrm{c} + 1}{a_1^2} A_1 + \frac{\sigma_\mathrm{c}^2}{2}(A_3 - A_2) + \frac{1}{a_2^2}\left([(a_2\sigma_\mathrm{c} + 1)q^{-1} - 1]B_2 + [(a_2\sigma_\mathrm{c} - 1)q + 1]B_3\right) + \frac{1 - a_4\sigma_\mathrm{c}}{a_4^2} A_4 \tag{33}$$

B Stress redistribution induced by a plastic strain

We follow the derivation of Ref. [2]. We decompose the strain in an elastic part and a localized plastic part:

$$\boldsymbol{\epsilon}(\boldsymbol{r}) = \boldsymbol{\epsilon}^\mathrm{el}(\boldsymbol{r}) + \boldsymbol{\epsilon}^\mathrm{pl}\delta(\boldsymbol{r}), \tag{34}$$

where

$$\boldsymbol{\epsilon}^\mathrm{pl} = \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \tag{35}$$

we work in dimension $d = 2$. The strain is related to the displacement field $\boldsymbol{u}(\boldsymbol{r})$ through

$$\epsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right). \tag{36}$$

The elastic strain only induces a stress, which is given by Hooke's law:

$$\sigma_{ij} = 2\mu\epsilon_{ij}^\mathrm{el} + \lambda\epsilon_{kk}^\mathrm{el}\delta_{ij}, \tag{37}$$

where μ and λ are the Lamé coefficients; they are related to the Young modulus E and Poisson ratio ν by

$$\mu = \frac{E}{2(1+\nu)}, \tag{38}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{39}$$

The strain satisfies mechanical equilibrium:

$$\partial_i \sigma_{ij} = 0. \tag{40}$$

We determine the displacement field in Fourier space, denoting the wavevector $\boldsymbol{q}$. Mechanical equilibrium reads $q_i\tilde{\sigma}_{ij} = 0$, hence

$$q_i(2\mu\tilde{\epsilon}_{ij} + \lambda\tilde{\epsilon}_{kk}\delta_{ij}) = 2\mu q_i \epsilon^{\mathrm{pl}}_{ij}, \tag{41}$$

where we have used that the plastic strain is trace free, $\epsilon^{\mathrm{pl}}_{kk} = 0$; note that for the right hand side the Fourier transform is a constant since the plastic strain is localized. The strain field is given by $\tilde{\epsilon}_{ij} = \mathrm{i}(q_i\tilde{u}_j + q_j\tilde{u}_i)/2$. The equation for the displacement is thus

$$\mu q^2\tilde{u}_j + (\mu+\lambda)q_j q_i u_i = 2\mathrm{i}\mu q_i\epsilon^{\mathrm{pl}}_{ij}. \tag{42}$$

Multiplying by q_j, we get

$$q_i\tilde{u}_i = \frac{2\mathrm{i}\mu}{2\mu+\lambda}\frac{q_i q_j}{q^2}\epsilon^{\mathrm{pl}}_{ij}, \tag{43}$$

hence

$$\tilde{u}_j = \frac{2\mathrm{i}}{q^2}\left(q_k\epsilon^{\mathrm{pl}}_{jk} - \frac{\mu+\lambda}{2\mu+\lambda}\frac{q_j q_k q_l}{q^2}\epsilon^{\mathrm{pl}}_{kl}\right). \tag{44}$$

For the displacement we compute the strain,

$$\tilde{\epsilon}_{ij} = -\frac{1}{q^2}\left(q_i q_k\epsilon^{\mathrm{pl}}_{jk} + q_j q_k\epsilon^{\mathrm{pl}}_{ik} - \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{q_i q_j q_k q_l}{q^2}\epsilon^{\mathrm{pl}}_{kl}\right) = -\left(Q_{ik}\delta_{jl} + Q_{jk}\delta_{il} - \frac{2(\mu+\lambda)}{2\mu+\lambda}Q_{ij}Q_{kl}\right)\epsilon^{\mathrm{pl}}_{kl}, \tag{45}$$

where $Q_{ij} = q_i q_j/q^2$.

We can now compute the stress. The elastic strain is

$$\tilde{\epsilon}^{\mathrm{el}}_{ij} = -\left(\delta_{ik}\delta_{jl} + Q_{ik}\delta_{jl} + Q_{jk}\delta_{il} - \frac{2(\mu+\lambda)}{2\mu+\lambda}Q_{ij}Q_{kl}\right)\epsilon^{\mathrm{pl}}_{kl}. \tag{46}$$

From the Hooke's law, we get the stress

$$\tilde{\sigma}_{ij} = \left[-2\mu\left(\delta_{ik}\delta_{jl} + Q_{ik}\delta_{jl} + Q_{jk}\delta_{il} - \frac{2(\mu+\lambda)}{2\mu+\lambda}Q_{ij}Q_{kl}\right) - \frac{\lambda\mu}{2\mu+\lambda}\delta_{ij}Q_{kl}\right]\epsilon^{\mathrm{pl}}_{kl}. \tag{47}$$

Usually, elasto-plastic models focus on the shear stress σ_{xy}, which is given by

$$\tilde{\sigma}_{xy} = -4\mu\gamma\left[1 - \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{q_x^2 q_y^2}{q^4}\right]. \tag{48}$$

The constant term gives a Dirac in Fourier space; the other part is

$$\tilde{\sigma}_{xy} = \frac{8\mu(\mu+\lambda)}{2\mu+\lambda}\frac{q_x^2 q_y^2}{q^4}. \tag{49}$$

In real space it is

$$\sigma_{xy}(\boldsymbol{r}) = \frac{8\mu(\mu+\lambda)}{2\mu+\lambda}\int\frac{q_x^2 q_y^2}{q^4}\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^2} = \frac{8\mu(\mu+\lambda)}{2\mu+\lambda}\partial_x^2\partial_y^2 g(r), \tag{50}$$

where $g(r)$ is the Green function of the bi-harmonic equation, which is $g(r) = r^2[\log(r) - 1]/8\pi$ for $d = 2$. Taking the derivatives, we find

$$\partial_x^2\partial_y^2 g(r) = \frac{8\cos(\theta)^2\sin(\theta)^2 - 1}{4\pi r^2} = -\frac{\cos(4\theta)}{4\pi r^2}. \tag{51}$$

C Eshelby calculation

We consider a linear elastic material with Young's modulus E and Poisson's ratio ν. We assume that a sphere with radius a is submitted to a strain

$$\epsilon^s_{ij} = \gamma\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{52}$$

meaning that a displacement

$$u_i = \epsilon^s_{ij} x_j \tag{53}$$

is imposed at the surface of the sphere. Here we compute the deformation of the medium; this calculation is related to the more general calculation presented in Ref. [3].

The stress is related to the strain through the Hooke's law

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right), \tag{54}$$

and, at equilibrium, it satisfies

$$\partial_i \sigma_{ij} = 0. \tag{55}$$

We deduce that the equation for the displacement is

$$\partial_i \partial_i u_j + \frac{1}{1-2\nu}\partial_j \partial_i u_i = 0. \tag{56}$$

Looking for the solution under the form

$$u_i = \lambda_1 \epsilon^s_{ij} \partial_j \left(\frac{1}{r}\right) + \lambda_2 \epsilon^s_{jk} \partial_i \partial_j \partial_k (r) + \lambda_3 \epsilon^s_{jk} \partial_i \partial_j \partial_k \left(\frac{1}{r}\right), \tag{57}$$

we find that

$$u_i = \frac{a^3}{2(4-5\nu)}\left[5(1-2\nu) + 3\frac{a^2}{r^2}\right]\epsilon^s_{ij}\frac{x_j}{r^3} + \frac{15a^3}{4(4-5\nu)}\left(1-\frac{a^2}{r^2}\right)\epsilon^s_{jk}\frac{x_i x_j x_k}{r^5}. \tag{58}$$

For $r \gg a$, it simplifies to

$$u_i^{r\gg a} = \frac{5(1-2\nu)a^3}{2(4-5\nu)}\epsilon^s_{ij}\frac{x_j}{r^3} + \frac{15a^3}{4(4-5\nu)}\epsilon^s_{jk}\frac{x_i x_j x_k}{r^5}. \tag{59}$$

References

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