ICFP – Soft Matter
Microrheology – Solution

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1 Velocity correlations in Laplace space

1. At equilibrium, equipartition imposes:

$$\langle v_0^2\rangle = \frac{T}{m}. \tag{1}$$

2. The Laplace transform of the Langevin equation starting at $t=0$ is:

$$m[s\hat{v}(s) - v_0] = -\hat{\zeta}(s)\hat{v}(s) + \hat{\eta}(s). \tag{2}$$

so that the Laplace transform of the velocity is

$$\hat{v}(s) = \frac{mv_0 + \hat{\eta}(s)}{ms + \hat{\zeta}(s)}. \tag{3}$$

3. Multiplying by v_0 and averaging (over v_0 and η) leads to

$$\hat{C}(s) = \frac{T}{ms + \hat{\zeta}(s)}, \tag{4}$$

where we have used the equipartition (Eq. (1)).

4. Thus the complex modulus can be obtained as

$$\hat{G}(s) = s\hat{\eta}(s) = \frac{s\hat{\zeta}(s)}{6\pi a} = \frac{s}{6\pi a}\left[\frac{T}{\hat{C}(s)} - ms\right]. \tag{5}$$

2 Stationarity condition and correlation function of the noise

5. The double Laplace transform of the correlation $\mathcal{C}(t,t') = \langle v(t)v(t')\rangle$ is simply $\hat{\mathcal{C}}(s,s') = \langle \hat{v}(s)\hat{v}(s')\rangle$, so using Eq. (3) and averaging gives

$$\hat{\mathcal{C}}(s,s') = \frac{Tm + \hat{\mathcal{N}}(s,s')}{[ms+\hat{\zeta}(s)][ms'+\hat{\zeta}(s')]}, \tag{6}$$

6. The double Laplace transform of $\mathcal{N}(t,t')$ is

$$\hat{\mathcal{N}}(s,s') = \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}t' \mathrm{e}^{-st-s't'}\mathcal{N}(t,t'). \tag{7}$$

To use the stationarity, we decompose

$$\begin{aligned}
\hat{\mathcal{N}}(s,s') &= \int_{t>t'} \mathrm{d}t'\mathrm{e}^{-st-s't'}\mathcal{N}(t,t') + \int_{t<t'} \mathrm{d}t'\mathrm{e}^{-st-s't'}\mathcal{N}(t,t') && (8)\\
&= \int_0^\infty \mathrm{d}t' \int_0^\infty \mathrm{d}u\mathrm{e}^{-(s+s')t'-us}\mathcal{N}(t'+u,t') + \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}u\mathrm{e}^{-(s+s')t-us'}\mathcal{N}(t,t+u) && (9)\\
&= \int_0^\infty \mathrm{d}t' \int_0^\infty \mathrm{d}u\mathrm{e}^{-(s+s')t'-us}N(u) + \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}u\mathrm{e}^{-(s+s')t-us'}N(u) && (10)\\
&= \frac{\hat{N}(s)+\hat{N}(s')}{s+s'}. && (11)
\end{aligned}$$

We have used the parity of the correlation, $N(t) = N(-t)$.

Using this relation in Eq. (6), we get

$$\hat{\mathcal{C}}(s,s') = \frac{Tm(s+s') + \hat{N}(s) + \hat{N}(s')}{(s+s')[ms+\hat{\zeta}(s)][ms'+\hat{\zeta}(s')]}. \tag{12}$$

7. The calculation of the previous question is very general, and holds for any stationary function; hence it also applies to the correlation $\mathcal{C}(t,t')$, which should be stationary, hence

$$\hat{\mathcal{C}}(s,s') = \frac{\hat{C}(s) + \hat{C}(s')}{s+s'}. \tag{13}$$

Inserting Eq. (4) leads to

$$\hat{\mathcal{C}}(s,s') = \frac{Tm(s+s') + T\left[\hat{\zeta}(s) + \hat{\zeta}(s')\right]}{(s+s')[ms+\hat{\zeta}(s)][ms'+\hat{\zeta}(s')]}. \tag{14}$$

8. Comparing Eqs. (12) and (13), we obtain

$$\hat{N}(s) = T\hat{\zeta}(s), \tag{15}$$

meaning that

$$N(t-t') = \langle \eta(t)\eta(t') \rangle = T\zeta(|t-t'|). \tag{16}$$

This is a fluctuation-dissipation relation since it relates the correlations of the noise $N(t)$ (the fluctuations) to the friction $\zeta(t)$ (dissipation).

We have shown that it is legitimate to consider the process starting at $t = 0$ with an initial condition uncorrelated with the noise.

References

[1] T. G. Mason and D. A. Weitz. Optical Measurements of Frequency-Dependent Linear Viscoelastic Moduli of Complex Fluids. *Phys. Rev. Lett.*, 74(7):1250–1253, Feb 1995.

[2] M. Medina-Noyola and J. L. Del Rio-Correa. The fluctuation-dissipation theorem for non-Markov processes and their contractions: The role of the stationarity condition. *Physica A: Statistical Mechanics and its Applications*, 146(3):483–505, 1987.