ICFP – Soft Matter

Linear rheology of a suspension of swimmers

Anke Lindner, Vincent Démery

We determine the effective viscosity of a suspension of swimmers. First, we determine the flow created by a single swimmer [1, 2]. Then, we determine the active stress generated by a finite density of swimmers. Finally, we use the alignment of the swimmers with the flow to compute the effective viscosity of a suspension of swimmers [3], which can be measured experimentally for two different kind of swimmers [4, 5].

Technical note: questions requiring calculations are indicated with asterisks: no asterisk for less than three lines of calculations, one for less than 10 lines, and two for longer calculations.

1 Hydrodynamic flow around a swimmer

1. * The incompressible flow $\boldsymbol{u}(\boldsymbol{x})$ around an object submitted to a force $\boldsymbol{F}$ can be obtained by solving the Stokes equation and the incompressibility condition:

$$\eta\nabla^2\boldsymbol{u}(\boldsymbol{x}) = \nabla P(\boldsymbol{x}) - \boldsymbol{F}\delta(\boldsymbol{x}), \tag{1}$$

$$\nabla\cdot\boldsymbol{u}(\boldsymbol{x}) = 0. \tag{2}$$

The viscosity of water is $\eta \simeq 0.001\,\mathrm{Pa\,s}$. By writting these equations in Fourier space, show that the solution reads

$$u_\mu(\boldsymbol{x}) = O_{\mu\nu}(\boldsymbol{x})F_\nu, \tag{3}$$

where $O(\boldsymbol{x})$ is the Oseen tensor, which reads, in Fourier and real spaces

$$\tilde{O}_{\mu\nu}(\boldsymbol{k}) = \frac{1}{\eta k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), \tag{4}$$

$$O_{\mu\nu}(\boldsymbol{x}) = \int \tilde{O}_{\mu\nu}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d} = \frac{1}{8\pi\eta r}\left(\delta_{\mu\nu} + \frac{x_\mu x_\nu}{r^2}\right); \tag{5}$$

for a vector $\boldsymbol{x}$, we denote its components x_μ, and $r = |\boldsymbol{x}|$ and $k = |\boldsymbol{k}|$. You do not have to prove Eq. (5).

2. A Chlamydomona is around $10\,\mu\mathrm{m}$ in diameter, and its mass density is 5% more than that of water. What is the gravitational force on a cell? What is its sedimentation velocity? Compare it to its swimming velocity, around $v_0 \simeq 100\,\mu\mathrm{m/s}$.

3. A swimmer cannot exert a net force on the surrounding fluid: the propulsion force exerted by the flagella is balanced by the drag force on the cell body. The simplest hydrodynamic model for a swimmer is thus a force dipole. Using scaling arguments, determine the flow field that dominates (the one due to sedimentation or the one due to propulsion) as a function of the distance to the swimmer.

4. ** What is the flow field created by a force dipole? What is different if the propelling organ is in front of (for a "puller") or at the rear ("pusher") of the cell body? Compare this prediction to the flows around a Chlamydomona and an Escherichia coli given on Fig. 1.

2 Active stress generated by the swimmers

The Stokes equation with an active stress $\boldsymbol{\sigma}^{\mathrm{a}}(\boldsymbol{x})$ reads

$$\eta\nabla^2\boldsymbol{u}(\boldsymbol{x}) = \nabla P(\boldsymbol{x}) - \nabla\cdot\boldsymbol{\sigma}^{\mathrm{a}}(\boldsymbol{x}). \tag{6}$$

5. Show that the trace of the active stress tensor can be absorbed in the pressure, so that we can assume that the active stress is traceless.

6. * What is the stress tensor associated to a force dipole of direction $\hat{\boldsymbol{n}}$? You can write the density of forces, which is the divergence of the stress tensor. Express the traceless part of this stress tensor as a function of the tensor $q_{\mu\nu} = n_\mu n_\nu - \delta_{\mu\nu}/3$.

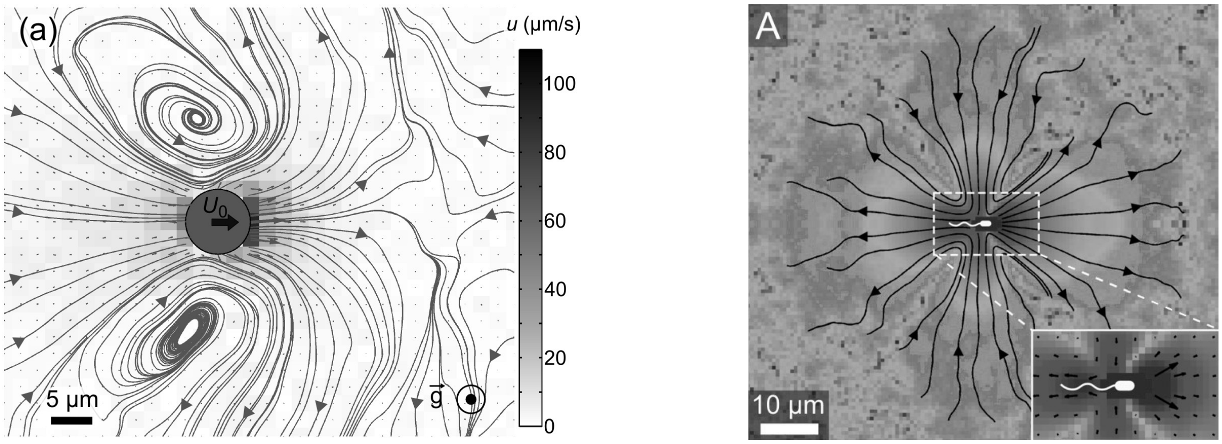

Figure 1: *Left:* flow around a chlamydomona [1]. *Right:* flow around an e-coli [2].

7. When there are many swimmers, at positions $\boldsymbol{x}_i$ and with orientations $\hat{\boldsymbol{n}}^i$, we define the local density $\rho(\boldsymbol{x})$ and nematic order parameter $\boldsymbol{Q}(\boldsymbol{x})$ as

$$\rho(\boldsymbol{x}) = \sum_i \delta(\boldsymbol{x} - \boldsymbol{x}_i), \tag{7}$$

$$\rho(\boldsymbol{x}) Q_{\mu\nu}(\boldsymbol{x}) = \sum_i \left(n^i_\mu n^i_\nu - \frac{\delta_{\mu\nu}}{3} \right) \delta(\boldsymbol{x} - \boldsymbol{x}_i). \tag{8}$$

What is the active stress field $\boldsymbol{\sigma}^{\mathrm{a}}(\boldsymbol{x})$ generated by this assembly of swimmers?

3 Effect on a shear flow

We assume that an external shear is imposed on the suspension of swimmers, so that the velocity field is given by $\boldsymbol{u}(\boldsymbol{x}) = 2\dot{\gamma} y \hat{\boldsymbol{e}}_x$. The tensorial shear rate is defined as $\dot{\epsilon}_{\mu\nu} = (\partial_\mu u_\nu + \partial_\nu u_\mu)/2$.

8. We assume that the order parameter and the density are uniform. The order parameter evolves according to

$$\partial_t Q_{\mu\nu} = -\frac{1}{\tau} Q_{\mu\nu} + \lambda \dot{\epsilon}_{\mu\nu}. \tag{9}$$

Discuss this equation. What is the order parameter in the stationnary regime?

9. The relation between the shear rate and the viscous shear stress $\boldsymbol{\sigma}^{\mathrm{v}}$ is given by $\sigma^{\mathrm{v}}_{\mu\nu} = 2\eta\dot{\epsilon}_{\mu\nu}$. The same relation with the total stress, $\boldsymbol{\sigma}^{\mathrm{v}} + \boldsymbol{\sigma}^{\mathrm{a}}$, defines the *effective viscosity* η_{eff}. What is the effective viscosity of the suspension of swimmers? Discuss the effect of the type of swimmers, pushers or pullers.

10. Explain qualitatively this behavior by drawing the flow created by a swimmer "aligned" with the imposed shear.

References

[1] Knut Drescher, Raymond E. Goldstein, Nicolas Michel, Marco Polin, and Idan Tuval. Direct Measurement of the Flow Field around Swimming Microorganisms. *Phys. Rev. Lett.*, 105(16):168101, Oct 2010.

[2] Knut Drescher, Jörn Dunkel, Luis H. Cisneros, Sujoy Ganguly, and Raymond E. Goldstein. Fluid dynamics and noise in bacterial cell–cell and cell–surface scattering. *Proceedings of the National Academy of Sciences*, 108(27):10940–10945, 2011.

[3] Yashodhan Hatwalne, Sriram Ramaswamy, Madan Rao, and R. Aditi Simha. Rheology of Active-Particle Suspensions. *Phys. Rev. Lett.*, 92(11):118101, Mar 2004.

[4] Andrey Sokolov and Igor S. Aranson. Reduction of Viscosity in Suspension of Swimming Bacteria. *Phys. Rev. Lett.*, 103(14):148101, Sep 2009.

[5] Salima Rafaï, Levan Jibuti, and Philippe Peyla. Effective Viscosity of Microswimmer Suspensions. *Phys. Rev. Lett.*, 104(9):098102, Mar 2010.