ICFP – Soft Matter

Linear rheology of a suspension of swimmers – Solution

Anke Lindner, Vincent Démery

1 Hydrodynamic flow around a swimmer

1. In Fourier space, the Stokes equation and the incompressibility condition read

$$-\eta k^2\tilde{u}_\mu(\boldsymbol{k}) = \mathrm{i}k_\mu\tilde{P}(\boldsymbol{k}) - F_\mu, \tag{1}$$

$$k_\mu\tilde{u}_\mu(\boldsymbol{k}) = 0. \tag{2}$$

Multiplying the first equation by k_μ and using the incompressibility leads to

$$\tilde{P}(\boldsymbol{k}) = -\mathrm{i}k_\mu F_\mu/k^2. \tag{3}$$

The first equation can be used to obtain the flow:

$$\tilde{u}_\mu(\boldsymbol{k}) = \frac{1}{\eta k^2}\left(F_\mu - \frac{k_\mu k_\nu}{k^2}F_\nu\right) = \tilde{O}_{\mu\nu}(\boldsymbol{k})F_\nu. \tag{4}$$

Taking the inverse Fourier transform leads to the result

$$u_\mu(\boldsymbol{x}) = O_{\mu\nu}(\boldsymbol{x})F_\nu. \tag{5}$$

2. The gravitational force on a cell is

$$F_\mathrm{g} = v\Delta\rho g = \frac{4}{3}\pi(5\times 10^{-6})^3 \times (0.05\times 10^3)\times 10 \simeq 2.6\times 10^{-13}\,\mathrm{N}. \tag{6}$$

Its sedimentation velocity is

$$v_\mathrm{s} = \frac{F_\mathrm{g}}{6\pi\eta a} = \frac{2.6\times 10^{-13}}{6\pi\times 10^{-3}\times(5\times 10^{-6})} \simeq 3\,\mathrm{\mu m/s}. \tag{7}$$

We find that $v_\mathrm{s} \ll v_0$.

3. The flow field due to sedimentation scales as

$$u_\mathrm{s}(r) \sim O(r)F_\mathrm{g} \sim \frac{1}{\eta r}\times\eta a v_\mathrm{s} \sim \frac{a}{r}v_\mathrm{s}. \tag{8}$$

The force dipole is characterized by its amplitude $Fa = \eta a v_0 \times a = \eta a^2 v_0$, which creates a flow

$$u_\mathrm{d}(r) \sim \frac{1}{\eta r^2}\eta a^2 v_0 \sim \frac{a^2}{r^2}v_0; \tag{9}$$

we have used that the flow created by a dipole should scale as r^{-2}. The two flows have the same amplitude at a distance r^* such that

$$\frac{r^*}{a} \sim \frac{v_0}{v_s} \simeq 30. \tag{10}$$

The flow created by the dipole dominates for $r < r^*$, the flow due to the sedimentation dominates for $r > r^*$.

4. We denote $\boldsymbol{n}$ the orientation of the force dipole. To avoid a continuous rotation of the swimmer, the forces should be aligned with $\boldsymbol{n}$. If the point forces are separated by a distance a, there is a force $F\boldsymbol{n}$ exerted on the fluid at $a\boldsymbol{n}/2$, and a force $-F\boldsymbol{n}$ exerted at $-a\boldsymbol{n}/2$. The flow is thus

$$u_\mu(\boldsymbol{x}) = \left[O_{\mu\nu}\left(\boldsymbol{x} - \frac{a}{2}\boldsymbol{n}\right) - O_{\mu\nu}\left(\boldsymbol{x} + \frac{a}{2}\boldsymbol{n}\right)\right]Fn_\nu \simeq -Fan_\nu n_\lambda\partial_\lambda O_{\mu\nu}(\boldsymbol{x}). \tag{11}$$

We compute the derivative:

$$\partial_\lambda O_{\mu\nu}(\boldsymbol{x}) = \frac{1}{8\pi\eta}\left[-\frac{x_\lambda}{r^3}\left(\delta_{\mu\nu}+\frac{x_\mu x_\nu}{r^2}\right)+\frac{1}{r}\left(\frac{\delta_{\mu\lambda}x_\nu+\delta_{\nu\lambda}x_\mu}{r^2}-2\frac{x_\mu x_\nu x_\lambda}{r^4}\right)\right] \tag{12}$$

$$= \frac{1}{8\pi\eta r^3}\left(\delta_{\mu\lambda}x_\nu+\delta_{\nu\lambda}x_\mu-\delta_{\mu\nu}x_\lambda-3\frac{x_\mu x_\nu x_\lambda}{r^2}\right). \tag{13}$$

Contracting with $n_\nu n_\lambda$ leads to

$$u_\mu(\boldsymbol{x}) = -\frac{Fax_\mu}{8\pi\eta r^3}\left(1-3\frac{(\boldsymbol{x}\cdot\boldsymbol{n})^2}{r^2}\right) = \frac{3Fax_\mu}{8\pi\eta r^3}\left(\cos(\theta)^2-\frac{1}{3}\right), \tag{14}$$

where θ is the angle between $\boldsymbol{n}$ and $\boldsymbol{x}$.

For $F > 0$ (propelling organ at the rear, pusher, E. coli), the flow is oriented away from the swimmer at the front and at the rear, and towards the swimmer on the sides. For $F < 0$ (propelling organ at the front, puller, Chlamydomona), it is the opposite.

The measured flow around an E coli seems close to the flow created by a dipole. The flow around a Chlamydomona is more complex, which is due to the two flagella; in this case a better model could be three point forces.

2 Active stress generated by the swimmers

5. We can just write

$$\nabla P - \nabla\cdot\boldsymbol{\sigma}^{\mathrm{a}} = \nabla P - \nabla\cdot\left(\boldsymbol{\sigma}^{\mathrm{a}}-\frac{1}{3}\operatorname{Tr}(\boldsymbol{\sigma}^{\mathrm{a}})\mathbf{1}+\frac{1}{3}\operatorname{Tr}(\boldsymbol{\sigma}^{\mathrm{a}})\mathbf{1}\right) = \nabla\left(P-\frac{1}{3}\operatorname{Tr}(\boldsymbol{\sigma}^{\mathrm{a}})\right)-\nabla\cdot\left(\boldsymbol{\sigma}^{\mathrm{a}}-\frac{1}{3}\operatorname{Tr}(\boldsymbol{\sigma}^{\mathrm{a}})\mathbf{1}\right). \tag{15}$$

6. The density of forces $\boldsymbol{f}(\boldsymbol{x})$ is

$$f_\mu(x) = Fn_\mu\left[\delta\left(\boldsymbol{x}-\frac{a}{2}\boldsymbol{n}\right)-\delta\left(\boldsymbol{x}+\frac{a}{2}\boldsymbol{n}\right)\right] = -Fan_\mu n_\nu\partial_\nu\delta(\boldsymbol{x}) = \partial_\nu\sigma^{\mathrm{a}}_{\mu\nu}(\boldsymbol{x}). \tag{16}$$

We identify

$$\sigma^{\mathrm{a}}_{\mu\nu}(\boldsymbol{x}) = -Fan_\mu n_\nu\delta(\boldsymbol{x}). \tag{17}$$

The traceless part is

$$\tilde{\sigma}^{\mathrm{a}}_{\mu\nu}(\boldsymbol{x}) = -Fa\left(n_\mu n_\nu-\frac{1}{3}\delta_{\mu\nu}\right)\delta(\boldsymbol{x}) = -Faq_{\mu\nu}\delta(\boldsymbol{x}). \tag{18}$$

7. With a density of swimmers, the stress tensor is

$$\boldsymbol{\sigma}^{\mathrm{a}}(\boldsymbol{x}) = -Fa\rho(\boldsymbol{x})\boldsymbol{Q}_{\mu\nu}(\boldsymbol{x}). \tag{19}$$

3 Effect on a shear flow

8. The order parameter aligns with the shear rate and relaxes with a rate $1/\tau$ (which is the rotational diffusion coefficient). In the stationary regime,

$$Q_{\mu\nu} = \lambda\tau\dot{\epsilon}_{\mu\nu}. \tag{20}$$

9. The active stress is

$$\boldsymbol{\sigma}^{\mathrm{a}} = -Fa\rho\boldsymbol{Q} = -Fa\rho\lambda\tau\dot{\boldsymbol{\epsilon}}. \tag{21}$$

The total stress is thus

$$\boldsymbol{\sigma}^{\mathrm{v}}+\boldsymbol{\sigma}^{\mathrm{a}} = (2\eta-Fa\rho\lambda\tau)\,\dot{\boldsymbol{\epsilon}} = 2\eta_{\mathrm{eff}}\dot{\boldsymbol{\epsilon}}, \tag{22}$$

leading to

$$\eta_{\mathrm{eff}} = \eta-\frac{1}{2}Fa\rho\lambda\tau. \tag{23}$$

Pushers ($F > 0$) reduce the viscosity and pullers ($F < 0$) increase it. There can be a spontaneous flow if the effective viscosity becomes negative.

10. Pushers aligned with the flow seem to enhance the flow by their activity, while pullers reduce it.