ICFP – Soft Matter

Thermal Casimir effect

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Like the van der Waals interaction, the Casimir effect is a fluctuation induced force. Originally studied for quantum fluctuations [1], it can also arise from thermal fluctuations. There are two complementary points of view: the first is to compute the free energy of the electric field in the vaccuum confined between two conducting plates. The second is to compute the correlation between the density of charges between the two plates; this is the route that we take here. Recently, the Casimir effect has been studied in out-of-equilibrium situations [2, 3].

This calculation involves statistical physics and electrostatics, makes use of the Debye-Hückel approximation and the resulting interaction involves the Debye length in the two plates; these are tools commonly used in soft matter.

We consider two parallel conducting plates S_i, $i \in \{1, 2\}$, of area $A = \ell^2$ and separated by a distance L, each carrying a surface density $\bar{\rho}_i$ of mobile charge carriers with charge q_i [2]. We denote $\rho_i(\boldsymbol{x})$ the local surface density of the carriers in the plate i, where $\boldsymbol{x}$ denotes the location in the plate, and $z \in \{0, L\}$ is the transverse coordinate. We assume that each plate is neutral, the opposite charge carriers being immobile, hence there is no net electrostatic interaction between the plates. The charges interact via the Coulomb interaction, so that the electrostatic energy of two charges is given by $q_1 q_2/(4\pi\epsilon|\boldsymbol{r}_1 - \boldsymbol{r}_2|) = q_1 q_2 G(\boldsymbol{r}_1 - \boldsymbol{r}_2)$.

Technical note: questions requiring calculations are indicated with asterisks: no asterisk for less than three lines of calculations, one for less than 10 lines, and two for longer calculations.

1 Free energy and interaction force

The free energy of a configuration of the density fields is (taking $k_B = 1$):

$$F = \frac{1}{2}\sum_{i,j} q_i q_j \int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}' G(\boldsymbol{x} - \boldsymbol{x}', z_i - z_j)[\rho_i(\boldsymbol{x}) - \bar{\rho}_i][\rho_j(\boldsymbol{x}') - \bar{\rho}_j] + T\sum_i \int \mathrm{d}\boldsymbol{x}\rho_i(\boldsymbol{x})\log(\rho_i(\boldsymbol{x})). \tag{1}$$

1. Explain the nature of the different terms.

2. In a given configuration of the charges, the force f on the plate S_2, located at $z = L$, is the derivative of the free energy with respect to L. Express the force f.

3. Show that the average of the force depends on the density correlations $C_{12}(\boldsymbol{x} - \boldsymbol{x}') = \langle[\rho_1(\boldsymbol{x}) - \bar{\rho}_1][\rho_2(\boldsymbol{x}') - \bar{\rho}_2]\rangle$.

2 Correlations

In order to compute the correlations, we use the Debye-Hückel approximation and assume weak density fluctuations $n_i(\boldsymbol{x}) = \rho_i(\boldsymbol{x}) - \bar{\rho}_i$.

4. Expand the free energy (1) in powers of $n_i(\boldsymbol{x})$ and retain only the quadratic terms; this is the Debye-Hückel free energy, F_{DH}.

We will compute the correlations in Fourier space. We define

$$\tilde{n}_i(\boldsymbol{k}) = \frac{1}{2\pi}\int \mathrm{d}\boldsymbol{x}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} n_i(\boldsymbol{x}), \tag{2}$$

$$n_i(\boldsymbol{x}) = \frac{1}{2\pi}\int \mathrm{d}\boldsymbol{k}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{n}_i(\boldsymbol{k}). \tag{3}$$

The Fourier transform of the Green function is given by $\tilde{G}(\boldsymbol{k}, z) = \mathrm{e}^{-kz}/(4\pi\epsilon k)$.

5. * Write the Debye-Hückel free energy for the Fourier modes of the density fluctuations.

6. Write the free energy under the form

$$F_{\mathrm{DH}} = \frac{1}{2}\sum_{i,j}\int \mathrm{d}\boldsymbol{k}\Delta_{ij}(\boldsymbol{k})\tilde{n}_i(\boldsymbol{k})^*\tilde{n}_j(\boldsymbol{k}). \tag{4}$$

Express $\Delta_{ij}(\boldsymbol{k})$.

7. From equipartition, the correlation function is given by

$$\langle n_i(\boldsymbol{k})^*\tilde{n}_j(\boldsymbol{k})\rangle = 2\pi\tilde{C}_{ij}(\boldsymbol{k}) = T\Delta_{ij}^{-1}(\boldsymbol{k}), \tag{5}$$

where $\tilde{C}_{ij}(\boldsymbol{k})$ is the Fourier transform of $C_{ij}(\boldsymbol{x})$. Using that the inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, write down the correlation between the densities 1 and 2, $\tilde{C}_{ij}(\boldsymbol{k})$. You can introduce the Debye lengths $l_i = T\epsilon/(\bar{\rho}_i q_i^2)$; discuss the role of these lengths in the correlation.

3 Force

8. Write the expression for the force obtained in question 3 for the Fourier transforms $\tilde{G}(\boldsymbol{k}, z)$ and $\tilde{C}_{12}(\boldsymbol{k})$.

9. Using the correlation $\tilde{C}_{12}(\boldsymbol{k})$ computed at the question 7 in the expression above, give the average Casimir force between the two plates. Show that it is attractive.

10. * In the limit where the distance between the plates is much larger than the Debye lengths, compute the force and show that it is universal (it does not depend on the properties of the plates). We provide $\int_0^\infty u^2\mathrm{d}u/(\mathrm{e}^u - 1) = 2\zeta(3)$.

11. Using dimensional analysis, recover the universal scaling. Explain the scaling of the quantum Casimir effect, $f \propto Ahc/L^4$, where h is the Planck constant and c the speed of light in the vaccuum.

12. Compute the force in the opposite limit.

References

[1] Hendrik B. G. Casimir. On the attraction between two perfectly conducting plates. *Proceedings of the Royal Netherlands Academy of Arts and Sciences*, 51(7):793–795, 1948.

[2] David S. Dean and Rudolf Podgornik. Relaxation of the thermal Casimir force between net neutral plates containing Brownian charges. *Phys. Rev. E*, 89(3):032117, Mar 2014.

[3] David S. Dean, Bing-Sui Lu, A. C. Maggs, and Rudolf Podgornik. Nonequilibrium Tuning of the Thermal Casimir Effect. *Phys. Rev. Lett.*, 116(24):240602, Jun 2016.